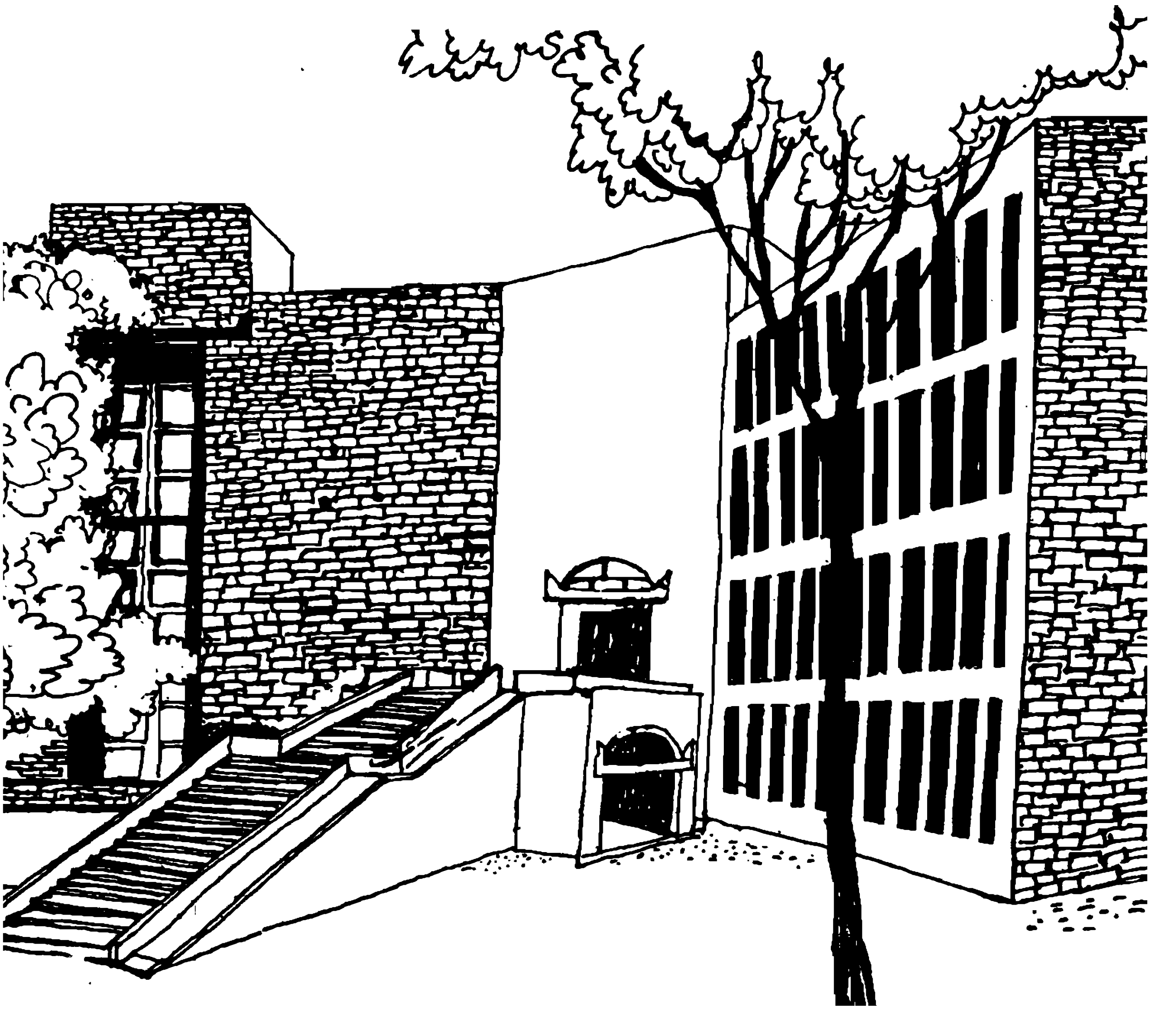
Working Paper



AN ELEMENTARY PROOF IN RATIONAL CHOICE THEORY REVISITED

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February 1997

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Abstract

In this paper we prove a result which, apart from having independent interest, has found applications in recent mathematical economic literature of rational choice theory. The result states that if a two-dimensional demand function satisfies budget exhaustion, the Weak Axiom of Revealed Preference and its range contains the strictly positive orthant of two dimensional Euclidean space, then it is representable by an utility function which is upper semicontinuous on the non-negative orthant of two dimensional Euclidean space and strictly quasi-concave and strictly monotonically increasing on the strictly positive orthant of two dimensional Euclidean space. By strictly monotonically increasing on the strictly positive orthant of two dimensional Euclidean space we mean that if a strictly positive vector is semi-strictly greater than another vector in the non-negative orthant of two dimensional Euclidean space, then the former has greater utility than the latter.

A two dimensional linear (competitive) budget problem is a set $S(p) = \{ x \in \mathbb{R}^2_+ / p.x \leq 1 \}$, for some $p \in \mathbb{R}^2_{++}$. Let C be the set of all two dimensional linear budget problems.

A choice function on C (also referred to as a demand function) is a function $f : C \rightarrow \mathbb{R}^2_+$ such that

$f(S(p)) \in S(p) \; \forall \, p \in \mathbb{R}^2_{++}$.

A demand function $f : C \rightarrow \mathbb{R}^2_+$ is said to satisfy the budget exhaustion property if $p.f(S(p) = 1 \; \forall \, p \in \mathbb{R}^2_{++}$.

Given a demand function $f: C \rightarrow \mathbb{R}^2_+$, define a binary relation R_f as follows: for $x, y \in \mathbb{R}^2_+$, $x R_f y$ if and only if $x \neq y$ and there exists $p \in \mathbb{R}^2_{++}$ such that x = f (S(p)) and $y \in S(p)$.

Given a demand function $f: C \to \mathbb{R}^2_+$, let $T(R_f)$ denote the transitive hull of R_f.

A demand function $f: C \to \mathbb{R}^2_+$ is said to satisfy

(i) the Weak Axiom of Revealed Preference (WARP) if R_f is asymmetric;

(ii) the Strong Axiom of Revealed Preference (SARP) if $T(R_f)$ is asymmetric.

Theorem 1 (Rose [1958]): Let $f: C \to \mathbb{R}^2_+$, be a demand function which satisfies the budget exhaustion property. Then f satisfies WARP if and only if f satisfies SARP.

A demand function $f: C \to \mathbb{R}^2_+$, is said to be representable by a utility function $V: \mathbb{R}^2_+ \to \mathbb{R}$ if $\forall\, p \in \mathbb{R}^2_{++}$,

$$\{ f(S(p)) \} = \{ x \in S(p) \,/\, V(x) \geq V(y) \;\forall\, y \in S(p) \}.$$

An utility function $V: \mathbb{R}^2_+ \to \mathbb{R}$ is said to be

(a) upper semicontinuous on $\mathbb{R}^2_+$ *if* $\forall c \in \mathbb{R}$, the set

$\{ x : V(x) \geq c \}$ is closed in $\mathbb{R}^2_+$

(b) strictly quasi-concave on $\mathbb{R}^2_{++}$ if $\forall x, y \in \mathbb{R}^2_{++}$ with

$x \neq y, \quad V(tx+(1-t)y) > \min \{ V(x), V(y) \} \; \forall \; t \in (0,1)$.

(c) monotonic on $\mathbb{R}^2_{++}$ if

$\forall \; x \in \mathbb{R}^2_{++}, y \in \mathbb{R}^2_+, \; x > y \; (i.e. \; x \geq y \; and \; x \neq y \;) \Rightarrow V(x) > V(y)$.

<u>Theorem 2</u>:- Let $f : C \Rightarrow \mathbb{R}^2_+$ be a demand function which satisfies the budget exhaustion property, WARP and suppose range $(f) \equiv \{ x \in \mathbb{R}^2_+ \; / \; \exists S(p) \in C$ with $f(S(p)) = x\}$ contains

$\mathbb{R}^2_{++}$. Then f is representable by an utility function V which is upper semicontinuous on $\mathbb{R}^2_+$ and strictly quasi-concave and monotonic on $\mathbb{R}^2_{++}$.

The proof of Theorem 2 relies on the following lemma:

Lemma 1 :- Let $f : C \Rightarrow \mathbb{R}^2_+$ be a demand function which satisfies all the conditions in Theorem 2. Let $x\, R_f\, y$. Then there exists $z \in \mathbb{R}^2_+$ and a neighbourhood ϑ of y such that $x\, R_f\, z$ and $z\, R_f\, s\ \forall\, s \in \vartheta$.

Proof of Lemma 1:- There are precisely two cases which can arise after noting that by budget exhaustion $x \neq 0$.

Case 1:- either $x_1\, y_2 \neq 0 \lor x_2\, y_1 \neq 0$

Case 2:- both $x_1\, y_2 = 0 \land y_1\, x_2 = 0$.

In Case 1, $z = \frac{1}{2}x + \frac{1}{2}y \in \mathbb{R}^2_{++} \subset range\ (f)$

Let $x = f(S(p))$ and $z = f(S(\bar{p})), p, \bar{p} \in \mathbb{R}^2_{++}$ with $y \in S(p)$.

Thus $z \in S(p)$. By WARP, $\bar{p}.x > 1$. Thus $\bar{p}.y < 1$ (or else, $\bar{p}.z > 1$ which is impossible).

Now, $x = f\ (S(p))$, $z \neq x$ and $z \in S(p) \Rightarrow x\ R_r\ z$.

Further $\bar{p}.y<1$ implies that there exists a neighbourhood ϑ of y such that $\bar{p}.s<1\ \forall\ s \in \vartheta$.

Thus $zR_r\ s\ \forall\ s \in \vartheta$.

In Case 2, if $x = f\ (S(p))$ then $p.y < 1$.

Let $p.y<\alpha<1 \wedge let\ \bar{p}=\frac{p}{\alpha}$. Let $z=f(S(\bar{p}))$.

Now, $p.z=\alpha<1 \Rightarrow xR_{r}z$.

Further $\bar{p}.y=\frac{p}{\alpha}.y<1$ implies there exists a neighbourhood

ϑ of y such that, $\bar{p}.s<1 \; \forall \; s \in \vartheta$.

Thus $z \; R_{r} \; s \; \forall \; s \in \vartheta$.

Q.E.D.

<u>Proof of Theorem 2</u>:- $\mathbb{R}^2_{+}$ has a countable neighbourhood base for its topology, for instance the set of open balls with rational radii and rational numbers for the coordinates of the centre.

Let $\{ \vartheta_n \}_{n \in N}$ be any such countable base. Define $v : \mathbb{R}^2_{+} \Rightarrow \mathbb{R}$ as follows: if $x \in \mathbb{R}^2_{+} \setminus$ range(f), put v(x) = -1; if x $\in$ range(f), let

$N(x) = \{ n \in \mathbb{N} \, / \, x \in \vartheta_n$ or there exists $w \in \vartheta_n$ with $WT(R_{r})\,x\}$; let $v(x) = \sum_{n \in N(x)} 2^{-n}$. . We will show that

$x R_t y \Rightarrow V(x) > V(y)$.

If $y \notin range(f)$, then $v(x) > 0 > -1 = v(y)$. If $y \in range(f)$, then $n \in N(x) \Rightarrow n \in N(y)$.

Hence $v(x) \geq v(y)$.

By Lemma 1, there exists ϑ_n such that $x \notin \vartheta_n$, $y \in \vartheta_n$ and $xT(R_t)\ s\ \forall s \in \vartheta_n$.

By Theorem 1 (and hence SARP), $s \in \vartheta_n \Rightarrow \neg\, s\ T(R_t)\ x$.

Thus there exists $n \in N(y) \setminus N(x)$.

Thus $v(x) > v(y)$.

Define $V : \mathbb{R}^2_+ \Rightarrow \mathbb{R}$ as follows:

$$V(x) = inf.\{ Sup\{ v(s) : s \in \vartheta \} / \vartheta \text{ is a neighbourhood of } x \}.$$

Let us check that V is upper semicontinuous.

Let $\{x^n\}_{n \in N}$ be a sequence in $\{x \in \mathbb{R}^2_+ / V(x) \geq c\}$. Let

$$\lim_{n \to \infty} x^n = \bar{x}.$$

Let ϑ be any neighbourhood of $\bar{x}$.

Since $\lim_{n \to \infty} x^n = \bar{x}$, there exists $N \in \mathbb{N}$ such that

$$x^n \in \vartheta \ \forall \ n \geq N.$$

$\therefore V(x^n) \geq c$ implies,

$$\sup. \{v(s) / s \in \vartheta\} \geq c.$$

This being true for every neighborhood of $\bar{x}$, we have,

$$V (\bar{x}) \geq c.$$

Suppose $x R_i y$. Hence by Lemma 1, there exists $z \in \mathbb{R}^2$ and a neighborhood ϑ of y such that $x R_i z$ and $z R_i s \ \forall \ s \in \vartheta$.

Thus, $v(x) > v(z) > v(s)\ \forall\ s \in \vartheta$

$\therefore v(x) > v(z) \geq \sup\{ v(s) / s \in \vartheta \}$.

$\therefore v(x) > v(z) \geq V(y)$.

Let $\overline{\vartheta}$ be any neighbourhood of x.

Then $\sup\{ v(s) / s \in \overline{\vartheta} \} \geq v(x) > v(z) \geq V(y)$

$\therefore V(x) \geq v(x) > v(z) \geq V(y)$.

Thus V (x) > V (y) and so V represents f.

Let $x \in \mathbb{R}^2_{++}$. Thus there exists $p \in \mathbb{R}^2_{++}$ such that x = f (S(p)).

Since $y \in \mathbb{R}^2_{-}$, $x > y$ implies $y \in S(p) \wedge y \neq x$, we have $x\ R_f\ y$. Thus V (x) > V (y) by the above.

Now let $x, y \in \mathbb{R}^2_{--}$, $x \neq y \wedge t \in (0,1)$. Suppose V (x) > V (y).

Let $z = tx + (1-t)y \wedge z = f(S(\overline{p}))$.

If $\bar{p}.x \leq 1, \bar{p}.y \leq 1$, *then* $z\, R_t\, x \wedge z\, R_t\, y$ so that V (z) > V (x) and

V (z) > V (y).

Suppose $\bar{p}.y > 1$. *Then* $\bar{p}.z = 1$ implies $\bar{p}.x < 1$, so that $z\, R_t\, x$

and so, V (z) > V (x).

If $\bar{p}.x > 1$, *then* $\bar{p}.y < 1$ and so V (z) > V (y).

Thus V (z) > min. { V (x), V (y) }.

Note $z \neq x$, $z \neq y$, by construction.

Q. E. D.

Discussion:- Theorem 2 has proved to be of crucial importance in research on rational choice theory that has been undertaken in recent years (see Bossert [1994], Lahiri [1997]). It is a result of some interest in its own right.

Bossert [1994], implicitly assumes this result by making a reference to Hurwicz and Richter [1971]. In Bossert [1994], it is stated that if x is revealed preferred to y (where both x and y are in the range of f), then the fact that x is revealed preferred to a set of points in the neighbourhood of y, would lead to Theorem 2 by methods indicated in Hurwicz and Richter [1971]. Although we do not dispute that it guarantees representability, it is not entirely obvious to us as to how without Lemma 1 (the crucial lemma in our paper), we could prove upper-semicontinuous representability. Our proof of Lemma 1 and Theorem 2, largely follows the elegant presentation in Sondermann [1982]. However, Sondermann's result does not imply our result. Sondermann requires the range of 'f' to satisfy a connectedness property which may not be satisfied under our assumptions. The variation in content and style though quite modest, is worth presenting in view of recent applications of Theorem 2.

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