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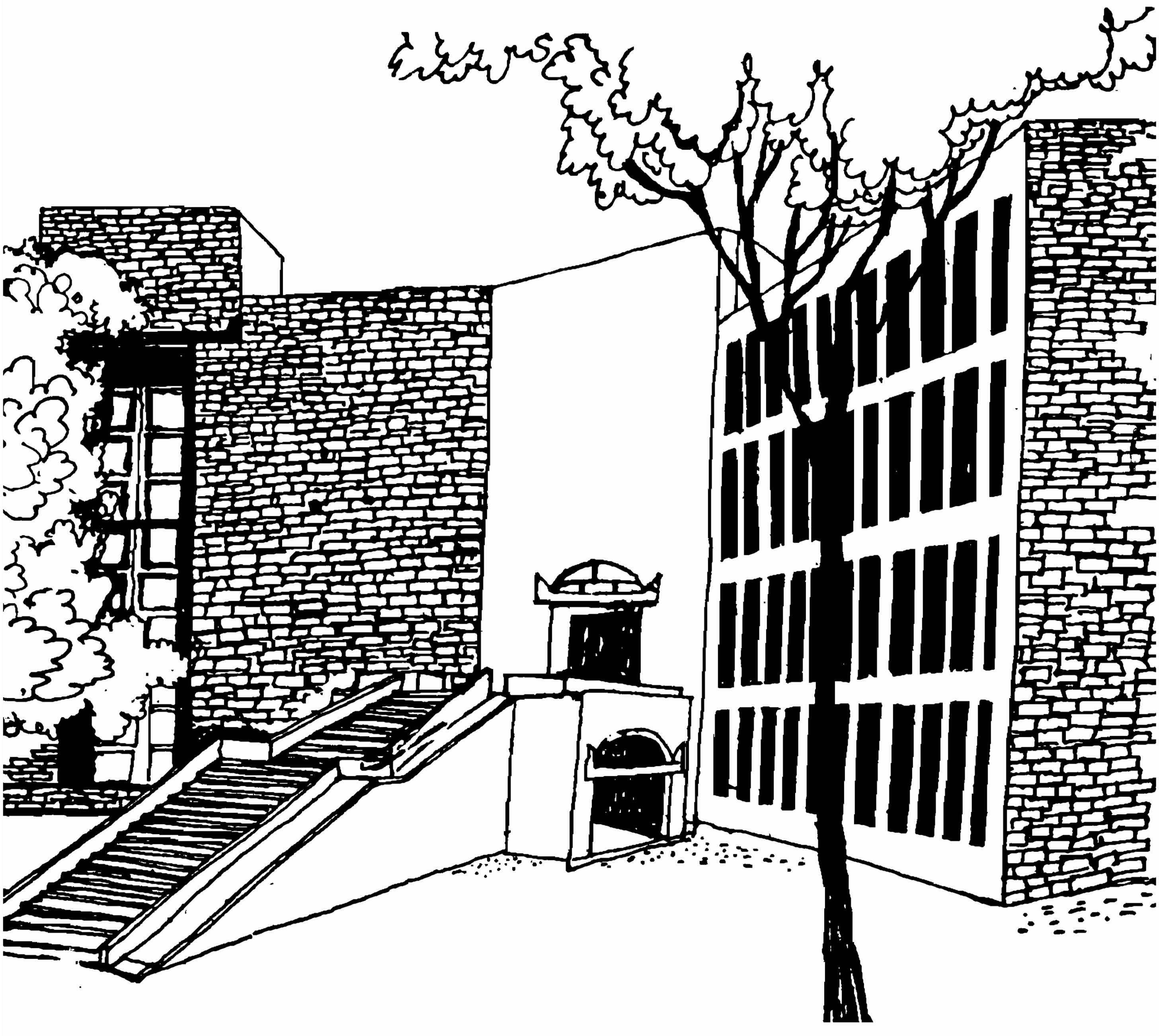

THE ORLOVSKY SOLUTION FOR ACYCLIC COMPARISON FUNCTIONS

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W.P.No. 2000-05-02
May 2000 1598

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ABSTRACT

In this paper we characterise the Orlovsky solution and what we refer to as the family of threshold solutions. Our family of threshold solutions are somewhat larger than the family of solutions which select only those alternatives that secure a certain prespecified proportion of the votes against all other alternatives.Our family of threshold solutions also include ones which select only those alternatives that exceed a certain prespecified proportion of the votes against all other alternatives.This is the price we pay for ommitting continuity from our axiomatic chracterizations.

Key Words:Fuzzy Relations,Decision analysis,Multiple criteria evaluation,Group decision-making.

Introduction: The gist of what is known as the Condorcet Paradox is the following phenomenon: given three alternatives x, y, z, it is possible to find rankings of these three alternatives such that a weak majority (i.e. at least half the total number of voters)prefer x to y,a weak majority prefer y to z and a weak majority prefer z to x.This obviously put the mathematical theory of electoral processes in a hopeless bind, since there does not exist in the above situation an alternative which is preferred to all others by a weak majority. However,it was also realised that in situations such as above, there was no obvious basis to discriminate among alternatives.

Subsequently the focus of voting theory shifted to enumerating the number of voters who preferred one alternative over another and selecting alternatives on the basis of such information.Although implicitly such a procedure has been popular for sometime now,a formal study of similar methods seems to be available for the first time in Arrow and Raynaud [1] in the guise of outranking matrices.These matrices simply tabulate the number of voters who prefer one alternative to another,with the entries being made appropriately in order to form a matrix.Subsequently Dutta and Laslier [7] refer to such procedures and call them comparison functions. The important thing to be noticed is that no analysis using compromise functions depends on any information other than the proportion of voters who prefer one alternative over another.The exact number of such voters is immaterial.Hence,in this paper we call the rule which assigns to each ordered pair of alternatives, the proportion of the voters who prefer one alternative to another,a comparison function.With this understanding a comparison function is simply what is known in the literature of fuzzy set theory, as a fuzzy binary relation or a valued binary relation (see Roubens [12]).

With decision rules now depending completely on comparison functions instead of, on individual preferences,there is clearly a need to reformulate voting theory.This has been done to a great deal in the paper by Dutta and Laslier [7] in situations,where an alternative which is preferred to all others by a weak majority does not exist.It is worth considering what the theory would look like if situations such as the Condorcet Paradox were prevented from arising by suitably restricting the domain.This is what we do in this paper.The solution to choice problems when preferences are such that they exclude the paradoxical situation mentioned above is what is known as the Orlovsky [10] solution in fuzzy set theory.

The axiomatic theory of choice rules with fuzzy preferences has a modest and yet rapidly growing literature as for instance [2], [3], [4], [5], [6], [8],[9], [10], [11], [13] and [14], to mention a few.Of particular interest are the axiomatic characterisations appearing in [14]of the Orlovsky solution and solutions which select only those alternatives that secure a certain prespecified proportion of the votes against all other alternatives.In this paper we axiomatically characterise these same solutions for compromise functions using almost the same assumptions that have been used by Sengupta in [14] except one,i.e.continuity.Continuity may be a meaningful assumption for fuzzy binary relations but is definitely not so if we restrict ourselves to compromise functions along with its intended interpretation as scores obtained in pairwise voting.Continuity would require the possibility of a comparison function to assume any value in the closed unit interval,where as such can never be the case if we consider proportions.Proportions such as those discussed above must always be rational numbers.While we do not exclude the possibility of other interpretations to our comparison functions,we do not deny it the possibility of representing an outranking matrix.Thus we have characterised the Orlovsky solution and what we call in this paper as the family of threshold solutions, using axioms similar to the ones in [14],but without using continuity at all.Our family of threshold solutions are somewhat larger than the family of solutions which select only those alternatives that secure a certain prespecified proportion of the votes against all other alternatives,as suggested by Sengupta in [14].Our family of threshold solutions also include ones which select only those alternatives that exceed a certain prespecified proportion of the votes against all other alternatives.This is the price we pay for ommitting continuity from our axiomatic chracterizations.It is worth pointing out that the entire analysis reported in this paper goes through if instead of considering real valued comparison functions, we considered only those that assumed values from among the set of rational numbers in the closed unit interval, as is likely to be the case in voting theory.

<u>The Framework</u>: Let $\mathbb{N}$ denote the set of natural numbers and let X be a non-empty finite set .Let [X] denote the set of all non-empty subsets of X.Let $D(X) \equiv \{(x,x)/x \in X\}$ denote the diagonal of X.Given a binary relation R on X (i.e.R is a subset of $X \times X$) let $P(R) \equiv \{(x,y) \in R/(y,x) \notin R\}$ be the asymmetric part of R and let $I(R) \equiv \{(x,y) \in R/(y,x) \in R\}$ be the symmetric part of R. A binary relation R on X is said to be:
reflexive if $D(X) \subset R$;
complete if $\forall (x,y) \in (X \times X) \backslash D(X)$:either $(x,y) \in R$ or $(y,x) \in R$;
transitive if $\forall x,y,z \in X$: $[(x,y),(y,z) \in R]$ implies $(x,z) \in R$;
quasi-transitive if P(R) is transitive;
acyclic if $\forall k \in \mathbb{N}$ and $x(1),\ldots,x(k) \in X$:$[(x(i),x(i+1)) \in P(R)\ \forall i \in \{1,\ldots,k-1\}]$ implies $[(x(k),x(1)) \notin P(R)]$.
Let G(X) denote the set of all reflexive and complete binary relations on X.Elements of G(X) are also referred to as abstract games.Let $U(X)=\{R \in G(X)/R \text{ is transitive}\}$, $Q(X)=\{R \in G(X)/R \text{ is quasi-transitive}\}$ and let $A(X)=\{R \in G(X)/R \text{ is acyclic}\}$. Given $(A,R) \in [X] \times G(X)$, let $B(A,R) = \{x \in A/ (x,y) \in R\ \forall y \in A\}$.B(A,R) is known as the set of best elements of R in A. The following result is well known:
<u>Proposition 1</u>: Given $R \in G(X)$: $[B(A,R) \neq \phi$ whenever $A \in [X]]$ if and only if $R \in A(X)$.

A <u>comparison function</u> g on X is a function $g: X \times X \rightarrow [0,1]$ such that $\forall x,y \in X: g(x,y)+g(y,x) = 1$.Hence $\forall x \in X: g(x,x) = 1/2$. Let $\Re$ denote the set of all comparison functions on X. Given $g \in \Re$, the binary relation $R(g) =\{(x,y) \in X \times X \ / \ g(x,y) \geq g(y,x)\}$ is called the Orlovsky relation generated by g. Clearly $R(g) \in G(X)$ whenever $g \in \Re$. Further if $R \in G(X)$ then $R = R(g)$, where $g \in \Re$ and is defined as follows:$g(x,y) = 1$ if $(x,y) \in P(R)$, $g(x,y) = 0$ if $(y,x) \in P(R)$ and $g(x,y) = 1/2$ if $(x,y) \in I(R)$.
Note:$(x,y) \in R(g)$ if and only if $[(x,y) \in X \times X$ and $g(x,y) \geq 1/2]$.
Let $H(A)=\{ g \in \Re/ R(g) \in A(X)\}$, $H(Q)=\{ g \in \Re/ R(g) \in Q(X)\}$ and $H(U)=\{ g \in \Re/ R(g) \in U(X)\}$. Following [14] we say that a comparison function g is <u>transitive</u> if $\forall x,y,z \in X$: $g(x,z) \geq \min \{g(x,y),g(x,z)\}$, and <u>weakly transitive</u> if $\forall x,y,z \in X$: $[(x,y),(y,z) \in R(g)]$ implies $[g(x,z) \geq \min \{g(x,y),g(x,z)\}]$.Let H(T) denote the set of all transitive comparison functions and

let H(W) denote the set of all weakly transitive comparison functions.Clearly, $H(T) \subset H(W)$ and $H(U) \subset H(Q) \subset H(A)$.

<u>Proposition 2</u>: $H(W) \subset H(U)$. Hence, $H(T) \subset H(W) \subset H(U) \subset H(Q) \subset H(A)$.

Proof: Let $g \in H(W)$ and suppose (x,y), $(y,z) \in R(g)$. Thus, $g(x,z) \geq \min\{g(x,y),g(x,z)\} \geq 1/2$. Thus $(x,z) \in R(g)$.Hence $g \in H(U)$. ♥

<u>Proposition 3</u>:Let $g \in H(U)$ and suppose $x,y,z \in X$.Then:

$[g(x,y)>g(y,x)]\&[g(y,z)\geq g(y,z)]$ implies $[g(x,z)>g(z,x)]$;

$[g(x,y)\geq g(y,x)]\&[g(y,z)>g(y,z)]$ implies $[g(x,z)>g(z,x)]$;

$[g(x,y)=g(y,x)]\&[g(y,z)=g(y,z)]$ implies $[g(x,z)=g(z,x)]$.

Proof: Follows obviously from the transitivity of R(g) whenever $g \in H(U)$. ♥

Let $\mathfrak{R}'$ be a non empty subset of $\mathfrak{R}$.A solution on $\mathfrak{R}'$ is a function $C:[X]\times \mathfrak{R}' \rightarrow [X]$ such that $\forall (A,g) \in [X]\times \mathfrak{R}'$: $C(A,g) \subset A$. $\mathfrak{R}'$ is called the domain of the solution C.

If C is a solution on $\mathfrak{R}'$ such that $\forall (A,g) \in [X]\times \mathfrak{R}'$: $C(A,g) \subset B(A,R(g))$, then C is said to be an <u>Orlovsky solution</u>.

By proposition 1,[$B(A,R(g)) \neq \phi$ whenever $(A,g) \in [X]\times \mathfrak{R}'$] if and only if $[R(g) \in H(A)]$.

Hence the domain of any Orlovsky solution must be contained in H(A).

Given $(A,g) \in [X]\times \mathfrak{R}'$ (:where $\phi \neq \mathfrak{R}' \subset \mathfrak{R}$)and $x \in A$, let $R(g,A,x)=\min\{g(x,y)/y \in A\}$.The following family of solutions is due to Sengupta ([14]): given $a \in [0,1/2]$ a solution $C:[X]\times \mathfrak{R}' \rightarrow [X]$ is said to be an a- threshold solution if $\forall (A,g) \in [X]\times \mathfrak{R}'$: $C(A,g) = M((A,g),a) \equiv \{x \in A / R(g,A,x) \geq a\}$.

The following proposition is easy to establish:

<u>Proposition 4</u>:$\forall a \in [0,1/2]$ and $\forall (A,g) \in [X]\times \mathfrak{R}'$: (i) $M((A,g),1/2)= B(A,R(g)) \subset M((A,g),a)$;(ii) if $a,b \in [0,1/2]$ with $a<b$, then $M((A,g),b) \subset M((A,g),a)$.

Proof: (i) It is easy to see that $\forall (A,g) \in [X]\times \mathfrak{R}'$, $M((A,g),1/2)= B(A,R(g))$.Hence let $(A,g) \in [X]\times \mathfrak{R}'$ and let $a \in [0,1/2]$.Suppose $x \in B(A,R(g))$.Thus,$R(g,A,x) \geq 1/2 \geq a$.Thus $x \in M((A,g),a)$.Thus, $B(A,R(g)) \subset M((A,g),a)$.

Let $x \in M((A,g),b)$ where $a,b \in [0,1/2]$ with $a<b$.Thus, $R(g,A,x) \geq b > a$.Hence, $x \in M((A,g),a)$.Thus, $M((A,g),b) \subset M((A,g),a)$. ♥

<u>Proposition 5</u>: Let $a \in [0,1/2]$ and $g \in H(A)$.Then $M((A,g),a) \neq \phi$ whenever $A \in [X]$.

Proof: It has been observed earlier, that as a consequence of proposition 1, $[M((A,g),1/2) \neq \phi$ whenever $A \in [X]]$ if and only if $g \in H(A)$.The proposition now follows from (i) of proposition 4. ♥

<u>Note</u>: It is not necessary for g to belong to H(A) in order to ensure the non emptiness of M((A,g),a) for A in [X],when a is less than 1/2.This is shown in the following example:

Example 1: Let $X=\{x,y,z\}$ with $x \neq y \neq z \neq x$ and let $g(x,x)=g(y,y)=g(z,z)=1/2, g(x,y)= g(y,z) =g(z,x) =3/4$, $g(y,x)= g(z,y) =g(x,z)=1/4$.Clearly, $(x,y),(y,z),(z,x) \in P(R(g))$.Hence $g \notin$. However $M((A,g),1/8) = A$ for all A in [X].

We now propose the following family of solutions which slightly modifies the family due to Sengupta [14]: given $a \in [0,1/2]$ a solution $C:[X]\times \mathfrak{R}' \rightarrow [X]$ is said to be a strict a - threshold solution if $\forall (A,g) \in [X]\times \mathfrak{R}'$: $C(A,g) =M^{+}((A,g),a) \equiv \{x \in A / R(g,A,x)>a\}$.

<u>Axioms For Solutions</u>: Let $C:[X]\times \mathfrak{R}' \rightarrow [X]$ be a solution.It is said to satisfy:

Chernoff's Axiom (CA) if $\forall (A,g),(B,g) \in [X]\times \mathfrak{R}'$: $[A \subset B]$ implies $[C(B,g) \cap A \subset C(A,g)]$;

Expansion (E) if $\forall$ (A,g),(B,g)$\in$ [X]$\times$ $\Re'$: C(B,g)$\cap$C(A,g) $\subset$ C(A $\cup$ B,g).
An interesting consequence of CA and E is the following:
Proposition 6: Let C:[X]$\times$ $\Re'$$\to$[X] be a solution satisfying CA and E .
(i)Suppose that for some g $\in$ $\Re'$ there exists a $\in$[0,1/2] such that [$\forall$x,y $\in$X: C({x,y},g) = M(({x,y},g),a)]. Then, $\forall$ A$\in$ [X]: C(A,g) = M((A,g),a).
(ii)Suppose that for some g $\in$ $\Re'$ there exists a $\in$[0,1/2] such that [$\forall$x,y $\in$X: M^+(({x,y},g),a) = C({x,y},g)]. Then, $\forall$ A$\in$ [X]: M^+((A,g),a) = C(A,g).
Proof :(i)Suppose that C:[X]$\times$ $\Re'$$\to$[X] be a solution satisfying CA and E and for some g $\in$ $\Re'$, there exists a $\in$[0,1/2],such that that $\forall$x,y $\in$X: C({x,y},g) = M(({x,y},g),a).Let, A$\in$[X]. Suppose x $\in$ C(A,g).By CA, [$\forall$y $\in$X: x $\in$C({x,y},g) = M(({x,y},g),a)].Thus [$\forall$y $\in$X: g(x,y)$\geq$a].Thus R(g,A,x)$\geq$ a. Thus, x $\in$ M((A,g),a).Hence, C(A,g)= M((A,g),a). Now suppose, x $\in$ M((A,g),a).Thus x $\in$ A and R(g,A,x) $\geq$ a. Thus, [x$\in$A]& [$\forall$y $\in$X: g(x,y)$\geq$ a]. Thus, [x$\in$A]& [$\forall$y $\in$X: x $\in$ M(({x,y},g),a) $\subset$ C({x,y},g)].By E, x$\in$C(A,g).Thus, M((A,g),a) $\subset$ C(A,g).Thus, C(A,g) = M((A,g),a). This proves (i).
is proved similarly. ♥
Let C:[X]$\times$ $\Re'$$\to$[X] be a solution.It is said to satisfy:
Neutrality (N) if (A,g),(B,h)$\in$[X]$\times$$\Re'$:[σ: A$\to$ B is one to one]&[$\forall$(x,y) $\in$ A$\times$A: g(x,y) = h(σ(x),σ(y))] implies [C(B,h) = {σ(x)/x$\in$C(A,g)];
Monotonicity (M) if $\forall$ x,y $\in$ X and g , h $\in$$\Re'$, if [h(x,y) $\geq$ g(x,y)] then :
x $\in$ C({x,y},g) implies x $\in$ C({x,y},h);
{x} = C({x,y},g) implies {x}= C({x,y},h).

Characterising the Threshold Solutions: A consequence of the above axioms is the following proposition:
Proposition 7: Let H(T) $\subset$ $\Re'$ $\subset$ H(A) and let C:[X]$\times$ $\Re'$$\to$[X] be a solution satisfying N and M . Then there exists a $\in$[0,1/2] such that : either (i)[$\forall$x,y $\in$X and $\forall$ g $\in$ $\Re'$: M^+(({x,y},g),a) = C({x,y},g)], or (ii)[$\forall$x,y $\in$X and $\forall$ g $\in$ $\Re'$: M(({x,y},g),a) = C({x,y},g)].
Proof : Let a = inf {g(x,y) / x $\in$ C({x,y}, g $\in$ $\Re'$ and x,y $\in$X}. Since the set {g(x,y) / x $\in$ C({x,y}, g $\in$ $\Re'$ and x,y $\in$X} is bounded below by zero, the infimum exists and is non negative. Since, g(x,x) = 1/2 for all x in X and g in $\Re'$,a $\leq$ 1/2. Let x,y $\in$X such that g(x,y) >a. Since a = inf {g(x,y) / x $\in$ C({x,y}, g $\in$ $\Re'$ and x,y $\in$X} , there exists x',y' $\in$X and h$\in$ $\Re'$ such that x' $\in$ C({x',y'}, h) and g(x,y)>h(x',y') > a. Let σ : X $\to$ X be defined by σ (x')=x,σ (y')=y, σ (x)=x',σ (y)=y' and σ (z)=z if z$\in$X\{x,y,x',y'} .Let f$\in$$\Re'$ be defined thus: f(σ (z),σ (w)) = h(z,w) whenever (z,w) $\in$X$\times$X. By N , x $\in$ C({x,y}, f). By M, x $\in$ C({x,y}, g).Thus, M^+(({x,y},g),a) $\subset$ C({x,y},g).
If there exists (x,y,g) $\in$ X$\times$X$\times$$\Re'$ such that g(x,y) < a and yet x $\in$ C({x,y}, g), then we would be contradicting our definition of a. Hence $\forall$ (x,y,g) $\in$ X$\times$X$\times$$\Re'$: x $\in$ C({x,y}, g) implies g(x,y) $\geq$ a, i.e. $\forall$ (x,y,g) $\in$ X$\times$X$\times$$\Re'$: C({x,y},g) $\subset$ M(({x,y},g),a) .Thus, $\forall$ (x,y,g) $\in$ X$\times$X$\times$$\Re'$: M^+(({x,y},g),a) $\subset$ C({x,y},g) $\subset$ M(({x,y},g),a).
Suppose that for some (x,y,g) $\in$ X$\times$X$\times$$\Re'$,g(x,y)=a and x $\in$ C({x,y}, g).By N, [$\forall$ (x,y,g) $\in$ X$\times$X$\times$$\Re'$: g(x,y)=a implies x $\in$ C({x,y}, g)].Combined with the fact [$\forall$ (x,y,g) $\in$ X$\times$X$\times$$\Re'$: M^+(({x,y},g),a) $\subset$ C({x,y},g) $\subset$ M(({x,y},g),a)], this yields the conclusion : [$\forall$x,y $\in$X and $\forall$ g $\in$ $\Re'$: M(({x,y},g),a) = C({x,y},g)].This proves the proposition. ♥

We are now in a position to state the following:
Theorem 1: Let $H(T) \subset \mathfrak{R}' \subset H(A)$ and let $C:[X] \times \mathfrak{R}' \to [X]$ be a solution on $\mathfrak{R}'$. Then C satisfies CA, E, N and M, if and only if there exists $a \in [0,1/2]$ such that
either(i)! $\forall (A,g) \in [X] \times \mathfrak{R}'$: $C(A,g) = M((A,g),a)$; or(ii) $\forall (A,g) \in [X] \times \mathfrak{R}'$: $C(A,g) = M^{+}((A,g),a)$.
Proof :It is easy to verify that if C is either an a-threshold solution or a strict a-threshold solution, then C satisfies CA, E, N and M.The converse assertion follows from propositions 6 and 7. ♥
However to obtain an axiomatic characterisation which uniquely characterises the Orlovsky solution we need a further axiom.

Axiomatic characterisation of the Orlovsky Solution : We now consider the following axiom:
Let $C:[X] \times \mathfrak{R}' \to [X]$ be a solution. It is said to be non trivial if whenever a compromise function g in $\mathfrak{R}'$ is non constant,then there exists $A \in [X]$ such that $C(A,g) \neq A$.
A non empty subset $\mathfrak{R}'$ of $\mathfrak{R}$ is said to be a rich domain if whenever a is a rational number in [0,1/2] then there exists $f,g \in \mathfrak{R}'$ and $x,y \in X: x \neq y$ and $g(x,y) = a = f(x,y)$.
Note : If $H(T) \subset \mathfrak{R}' \subset H(A)$ then $\mathfrak{R}'$ is a rich domain.
Proposition 8 :Let $\mathfrak{R}'$ be a rich domain and let $C:[X] \times \mathfrak{R}' \to [X]$ be a non trivial solution on $\mathfrak{R}'$ satisfying CA,E, N and M. Then, $\forall (A,g) \in [X] \times \mathfrak{R}'$: $C(A,g) = M((A,g),1/2)$.
Proof: Let $f,g \in \mathfrak{R}'$ and $x',y' \in X$: $x' \neq y'$ and $g(x',y') = 1/2 = f(x',y')$.By N, $C(\{x',y'\},g) = C(\{x',y'\},f) = \{x',y'\}$. Hence, by M, if $g \in \mathfrak{R}'$ and $g(x',y') \geq 1/2$, then $x' \in C(\{x',y'\},g)$.By N, if $(h,x,y) \in \mathfrak{R}' \times X \times X$ with $x \neq y$ and $h(x,y) \geq 1/2$, then $x \in C(\{x,y\},h)$.If $x = y$, then $h(x,y) = 1/2$ and $x \in C(\{x,y\},h)$.Thus,[$(h,x,y) \in \mathfrak{R}' \times X \times X$ with $x \neq y$ and $h(x,y) \geq 1/2$ implies $x \in \{x,y\},h)$.Thus $\forall h \in \mathfrak{R}'$:$(x,y) \in R(h)$ implies $x \in C(\{x,y\},h)$.
Now suppose that $g \in \mathfrak{R}'$ and towards a contradiction suppose that there exists $x',y' \in X$ such that $x' \in C(\{x',y'\},g)$ and yet $(x',y') \notin R(g)$. Thus $g(x,y) < 1/2$. Let a be any rational number strictly greater than g(x,y) and strictly less than 1/2.By M, if $h \in \mathfrak{R}'$ and $h(x',y') = a$, then $x' \in C(\{x',y'\},h)$. Further $1-a>a$ and the fact $(y',x') \in R(h)$ implies by the previous part that $y' \in C(\{x',y'\},h)$.Hence, if $h \in \mathfrak{R}'$ and $h(x',y') = a$, then $\{x',y'\} = C(\{x',y'\},h)$. Let $X = \{z(1),z(2),....,z(m)\}$ and $h \in \mathfrak{R}'$, with $h(z(i),z(j))=a$ if $i<j$, $h(z(i),z(j))=1/2$ if $i=j$, $h(z(i),z(j))=1-a$ if $i>j$.By N, $[\forall x,y \in X$: $C(\{x,y\},h) = \{x,y\}]$. Let $A \in [X]$. It follows as a consequence of E,that $C(A,h) = A$.Since h is non constant this contradicts the non triviality of C. Thus,$\forall h \in \mathfrak{R}'$: $[x,y \in X$ and $x \in C(\{x,y\},h)]$ implies $[(x,y) \in R(h)]$. In conjunction with what we have obtained earlier,it follows that $\forall h \in \mathfrak{R}'$: $[x,y \in X$ and $x \in C(\{x,y\},h)]$ if and only if $[(x,y) \in R(h)]$.The proposition now follows as a consequence of proposition 6. ♥

Theorem 2: Let $\mathfrak{R}'$ be a rich domain and let $C:[X] \times \mathfrak{R}' \to [X]$ be a non trivial solution on $\mathfrak{R}'$.Then C satisfies CA, E, N and M if and only if C is the Orlovsky solution on $\mathfrak{R}'$.
Proof :That the Orlovsky solution on $\mathfrak{R}'$ satisfies CA, E, N and M follows from theorem 1. The converse follows from proposition 8. ♥

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