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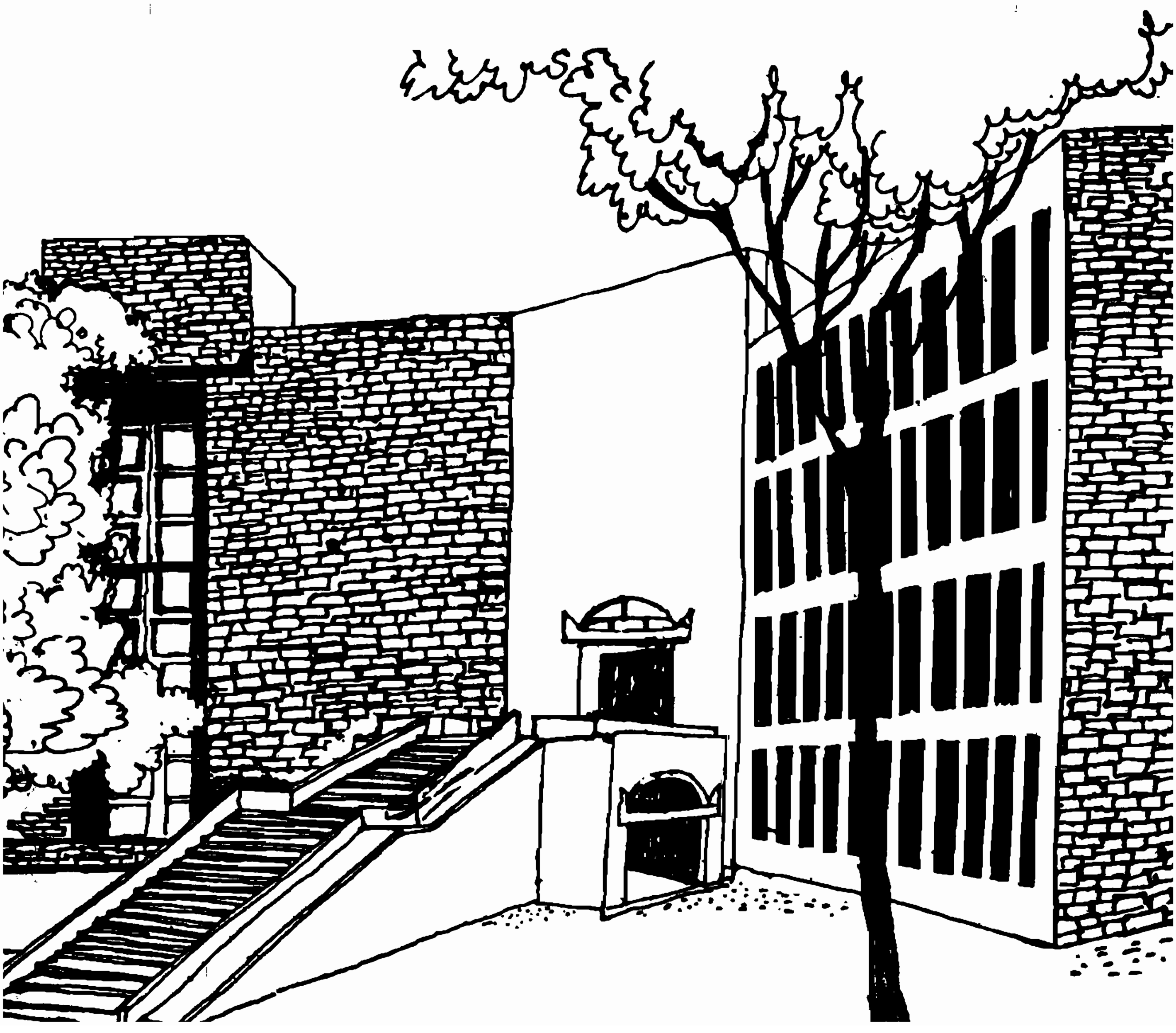

AXIOMATIC ANALYSIS OF CHOOSING THE SECOND BEST

By

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Introduction :

The dominant theme in decision theory has been the one where an agent chooses what is perceived to be the best outcome out of a (finite) set of outcomes.This has been the model that economic theory has traditionally favored.In a paper by Baigent and Gaertner (1996) we find a departure from this theme.It is argued there that if there is a unique best outcome then often one may forgo one's claim to it out of politeness. A similar consideration is that of altruism which manifests itself in similar behavior. However, can there be no other type of consideration which prompts one to judiciously avoid the best?

Consider a person who chooses an academic job over another which pays more salary and perks.Definitely if the criteria by which we evaluate choice in this kind of a situation is financial remuneration , then choice of a job with lower remuneration,violates the postulate of choosing the best.The person in question, by sacrificing some money probably wants to pursue a more mentally rewarding profession.One may say that the person was actually not maximizing monetary rewards and hence relating the person's behavior to a financial criteria is not appropriate.However, a similar kind of objection may be raised when a person surrenders his / her claim to a desirable object on grounds of politeness or altruism.Under such situations, acceptability by society or emotional satisfaction that comes out of giving to others and not financial rewards motivate human behavior.

There are religions in the world,notably Hinduism,which puts a premium on ascetic life styles and self denial. To my mind the purpose of such preachings is to highlight the fact that appearances at times may be deceptive and so we should be careful before we indulge our temptations.It is a different matter that to the popular mind such religions have become associated with exaggerated forms of atrocities perpetrated on oneself and suppression of natural desires.Anecdotes in religious literature, do emphasise the value of discipline in life and sometimes they are done in a way so that no one has any doubts about the real message that is meant to be conveyed.However the true purpose of such a philosophy is to prevail on us to use our judgment in decision making rather than fall for what may simply appear to be the best alternative without actually being so.Thus for instance a third peg of whisky may appear to be the best choice to a tippler although what is most appropriate under the circumstances is to call it a day.In such circumstances one forgoes one claim on what is apparently the best simply by harnessing one's temptations.It is not done out of considerations of politeness or social ettiquette but because the ranking of alternatives is recognized to be myopic. To convey the merits of self control to a tippler, one may have to narrate the virtues of abstinence when what one really advocates is moderation. In fact one may tempt in trying to deceive and hence an alternative may appear to be the best without actually being so.It is precisely because of such considerations that choosing the best should sometimes be consciously avoided.

An example of a decision procedure where the best alternative is not chosen is choosing the median. The median is a reasonable compromise,in practical decision making. In Gaertner and Xu (1999) can be found a first axiomatic characterisation of the choice rule which selects the median from a finite set of alternatives.The axiomatic characterisation is valid for a universal set containing at least four alternatives. For universal sets containing three alternatives the above mentioned axiomatic characterisation is no longer valid.However,decision theory as opposed to decision algorithms,has overriding importance only when the set of alternatives is sufficiently small.For large sets the computational complexity of a solution may substantially offset its decision theoretic virtues.For a set containing a small number of alternatives we may ignore computational issues and concentrate only on decision theoretic properties. In Lahiri (2000) we provide two theorems which characterizes the median choice function when the universal set has atleast three alternatives. As discussed in Lahiri (2000), choosing the median however turns out to be the result of a " menu-based optimization exercise".

In this paper we provide two axiomatic chracterizations of the decision rule which invariably selects the second best alternative. Unlike Baigent and Gaertner (1996) we restrict our selves to the situation where no two alternatives share the same rank.The work just cited is about selecting the second best alternative only if the there is a unique best alternative; otherwise the best alternative is selected.There the purpose was to characterize the behavior of an individual who wants to avoid the stigma of being labelled as " greedy".Our purpose on the other hand is to axiomatically analyze decision rules which invariably select second best alternatives. The relevant consideration here is not distancing oneself from the stigma attached to being greedy, but controlling desires and avoiding temptations so as not to fall prey to deceipt and consequent danger. We call our decision rules 'rank solutions', becuase choices are based on the perceived ranks of the alternatives and no other considerations.

<u>Rank Solutions</u>:

Let N denote the set of positive integers and let $X = \{i \in N / i \leq n\}$ (:the set of first n positive integers) for some $n \in N$ with $n \geq 3$.

A rank solution on X is a function $C:[X] \rightarrow [X]$such that $C(A) \subset A \ \forall A \in [X]$.

Let G: $[X] \rightarrow [X]$ be defined by $G(A) = \{i \in A / i \geq j, \forall j \in A\}$.G is known as the greatest rank solution .Clearly G is a single valued rank solution .Let $G(A) = \{g(A)\}$ whenever $A \in [X]$.

The second best rank solution on X is the function S: $[X] \rightarrow [X]$ defined as follows: $\forall A \in [X]$,(a) if $\#A \geq 2$ then $S(A) = G(A \backslash G(A))$; (b) if $\#A = 1$,then $S(A) = G(A)$.

Axioms :

The following axiom can be found in Lahiri (2000):
Axiom 1 : $\forall$ i,j $\in$ X with i $\neq$ j, if f : {i,j }$\mapsto$ X is one to one and order preserving (:in the sense that f (i) > f (j) if and only if i>j),then C({f (i),f (j)})= {f (k)/ k $\in$ C({i,j})}.
As in the work just cited, this axiom plays a crucial role in our axiomatic analysis.The next axiom is similar to an axiom appearingin Baigent and Gaertner (1996):
Axiom 2: $\forall$A$\in$[X] : if there exists i,j,k$\in$A,with i$\neq$j$\neq$k$\neq$i and i $\in$ C ({i,j}) $\cap$ C ({i,k}), then i $\notin$ C (A).
The next axiom can be found in Baigent and Gaertner (1996):
Axiom 3 : $\forall$A$\in$[X] with # A $\geq$ 2 : [x $\in$ C(A)] implies [there exists y $\in$ A \ {x} such that $\forall$ z$\in$ A \ {y}, z$\in$ C({y,z})].
The following axiom is being introduced in addition to the above so that a complete characterization of the second best rank solution is possible.
Axiom 4 : $\forall$i,j,k$\in$X,with i$\neq$j$\neq$k$\neq$i, [a > i whenever a $\in$ C ({i,j,k})] implies [C({i,j,k}) $\cap$ C({j,k}) $\neq$ ϕ].
It is easily verified that the second best rank solution S satisfies axioms 1 to 4. However the four axioms mentioned above are logically independent as is shown in the following examples where X = {1,2,3}.
Example 1: Let C({1,3}) = C({2,3}) = {3} and C(A) = {1} if #A $\geq$ 2, A $\neq$ {1,3} $\neq$ {2,3} $\neq$ A. Clearly C satisfies all the axioms mentioned above (and in particular axiom 4 vacuously) but does not satisfy axiom 1.
Example 2: The greatest rank solution G satisfies all the axioms except for axiom2, since {3} = C({1,2,3}) = C({1,3}) = C({2,3}).
Example 3: Let C({1,2,3}) = {1} and let C(A) = G(A) otherwise. C satisfies all the axioms mentioned above except for axiom 3.
Example 4: Let C({1,2,3}) ={2} and let C(A) = G(A) otherwise.C satisfies all the axioms except for axiom 4.
Hence we have proved the following:
Proposition 1 : S satisfies axioms 1 to 4.The four axioms are logically independent.
In Baigent and Gaertner (1996) the axiom similar to our axiom 2, that is actually used is the following:
Axiom 2' : $\forall$A$\in$[X] : if there exists i,j,k$\in$A,with i$\neq$j$\neq$k$\neq$i and {i} = C ({i,j}) = C ({i,k}), then i $\notin$ C (A).
Since axiom 2 implies axiom 2', S satisfies axiom 2'.However, S is not the only rank solution to satisfy Axioms 1,2',3,4 as the following example reveals:
Example 5 : Let C({1,2,3}) = {1} and let C(A) = A otherwise.C satisfies all the axioms and in particular it satisfies axiom 2' vacuously. However C $\neq$ S.
Axiom 3 is implied by the following:
Axiom 3' : $\forall$A$\in$[X] with # A $\geq$ 2 : [x $\in$ C(A)] implies [there exists y $\in$ A \ {x} such that $\forall$ z$\in$ A \ {y}, {z} = C({y,z})].

Clearly S satisfies axiom 3'. Observe that the rank solution in example 1 above satisfies axioms 2',3',4 but not axiom 1; the rank solution in example 2 satisfies axioms 1,3',4 but does not satisfy axiom 2'; the rank solution in example 3 satisfies axioms 1,2',4 but does not satisfy axiom 3'; the rank solution in example 4 satisfies axioms 1,2',3' but does not satisfy axiom 4.

Hence we have proved the following:

Proposition 2 : S satisfies axioms 1,2',3',4.The four axioms are logically independent.

The following proposition is revealing:

Proposition 3: Let C be a rank solution satisfying axiom1.Then C saisfies axioms 2 and 3 if and only if C satisfies axioms 2',3'.

Proof: Suppose C satisfies axiom 1.

(a) C satisfies axioms 2 and 3.

Hence C satisfies axiom 2'. Towards a contradiction suppose that C does not satisfy axiom 3'. Hence there exists $A \in [X]$ with $\# A \geq 2$ and there exists $x \in C(A)$ such that whenever $y \in A \setminus \{x\}$ there exists $z(y) \in A \setminus \{y\}$ and $\{y,z(y)\} = C(\{y,z(y)\})$. By axiom 1, for all i,j in X with $i \neq j$, $C(\{i,j\}) = \{i,j\}$. Let $i,j,k \in X$ with $i \neq j \neq k \neq i$ and suppose $i \in C(\{i,j,k\})$. This along with $i \in C(\{i,j\}) \cap C(\{i,k\})$ contradicts axiom 2. Thus $i \notin C(\{i,j,k\})$. By the same argument neither j nor k belongs to $C(\{i,j,k\})$ contradicting the non emptiness of $C(\{i,j,k\})$. Thus C satisfies axiom 3'.

(b) C satisfies axioms 2' and 3'.

Hence C satisfies axiom 3. Towards a contradiction suppose that C does not satisfy axiom 2. Hence there exists $A \in [X]$ and $i,j,k \in A$,with $i \neq j \neq k \neq i$ and $i \in C(\{i,j\}) \cap C(\{i,k\}) \cap C(A)$.By axiom 3', there exists $y \in A \setminus \{i\}$ such that $\forall z \in A \setminus \{y\}$, $\{z\} = C(\{y,z\})$.By axiom 1, $\{i\} = C(\{i,j\}) = C(\{i,k\})$. By axiom 2', we get $i \notin C(A)$ and a contradiction.Thus C satisfies axiom 3.

This proves the proposition.

In the above proposition the assumption that C satisfies axiom 1 is not superfluous as the following two examples reveal:

Let $X = \{1,2,3\}$.

Example 6 : Let $C(X) = \{1\}$, $C(\{1,2\}) = \{1\}$,$C(\{1,3\}) = \{1,3\}$,$C(\{2,3\}) =\{3\}$.C satisfies axiom 2' (vacuously) and it also satisfies axiom 3'.However C does not satisfy axiom 2, nor does it satisfy axiom1.

Example 7 : Let $C(X) = \{3\}$, $C(\{1,2\}) = \{1\}$,$C(\{1,3\}) = \{1\}$,$C(\{2,3\}) =\{2,3\}$.C satisfies axiom 2 and it also satisfies axiom 3.However C does not satisfy axiom 3', since there does not exist y in $\{1,2\}$ such that $C(\{y,z\}) = \{z\}$ whenever z belongs to $X \setminus \{y\}$.Note C does not satisfy axiom1 either.

It is worth noting that S does not satisfy the following property due to Nash (1950):

Nash's Independence of Irrelevant Alternatives (NIIA) : (a) $\forall A \in [X]$, $\# C(A) =1$; (b) $\forall A,B \in [X]$,with $A \subset B$,[$C(B) \subset C(A)$ implies $C(B) = C(A)$].

It is by now a standard result in choice theory that the satisfaction of NIIA by a rank solution C is equivalent to the existence of a function $u: X \rightarrow \Re$ (the set of real numbers) such that $\forall A \in [X]:C(A) = \{x \in A / \forall y \in A: u(x) \geq u(y)\}$ (see Aizerman and Aleskerov (1995) Theorem 2.10,for instance). Hence we can conclude that there does not exist a function $u: X \rightarrow \Re$ such that $\forall A \in [X]:S(A) = \{x \in A / \forall y \in A: u(x) \geq u(y)\}$.

<u>The main results</u> :

Theorem 1: The only rank solution on X to satisfy axioms 1,2,3 and 4 is S.
Theorem 2: The only rank solution on X to satisfy axioms 1,2',3' and 4 is S.
<u>Remark 1</u>: The assumption that $n \geq 3$ is crucial in what follows .If X ={1,2}, then C({i})={i} for all $i \in$ {1,2} and C({1,2})={2} satisfies all the properties mentioned above. However $C \neq S$.
The two theorems will be proved by appealing to a sequence of lemmas.
Lemma 1: Let C satisfy axioms 1 and 2. Then, $\forall A \in [X]$ with # A = 2, C(A) is a singleton.
Proof: Towards a contradiction suppose that for some i,j in X with $i \neq j$, C({i,j}) = {i,j}. Then by axiom 1, for all i,j in X with $i \neq j$, C({i,j}) = {i,j}. Let $i,j,k \in X$ with $i \neq j \neq k \neq i$ and suppose $i \in$ C({i,j,k}). This along with $i \in C(\{i,j\}) \cap C(\{i,k\})$ contradicts axiom 2. Thus $i \notin$ C({i,j,k}). By the same argument neither j nor k belongs to C({i,j,k}) contradicting the non emptiness of C({i,j,k}). This proves the lemma.
Lemma 2: Let C satisfy axioms 1,2 and 3. Then, $\forall A \in [X]$: C(A) is a singleton.
Proof: By lemma 1, C(A) is a singleton for $\forall A \in [X]$ with # A = 2. Hence suppose towards a contradiction that there exists $A \in [X]$ with # A > 2 for which $x,y \in C(A)$ and $x \neq y$. By axiom 3 and lemma 1, there exists $w \in A \setminus \{x\}$ and $u \in A \setminus \{x\}$ such that (a) {z} = C({z,w}) whenever z belongs to $A \setminus \{w\}$; (b) {z} = C({z,w}) whenever z belongs to $A \setminus \{w\}$. Hence {u} = C({u,w}) = {w}.Thus, u = w. Without loss of generality suppose x > y.
Case 1: x > u.
By axiom 1, C({x,u}) = {x} implies C({a,b}) = {a} if and only if $a \geq b$. Hence C({y,u}) = {y} implies that y > u. Thus, x > y > u.Thus, {x} = C({x,y}). By axiom 2, $x \notin C(A)$, contradicting our hypothesis that $x \in C(A)$.
Case 2 : u > x.
By axiom 1, C({x,u}) = {x} implies C({a,b}) = {a} if and only if $b \geq a$. Hence C({y,u}) = {y} implies that u > y. Thus, u > x > y.Thus, {y} = C({x,y}). By axiom 2, $y \notin C(A)$, contradicting our hypothesis that $y \in C(A)$.
Thus C(A) must be a singleton. This proves the lemma.
Lemma 3 : Let C be a rank solution on X which satisfies axioms 1,2,3 and 4. Then for all A in [X], C(A) = S(A).
Proof : Step 1: Claim : If $a, b \in X$ with a < b then C({a,b}) = {a}.
Proof of Claim: Let $x,y,z \in X$ with x < y < z and towards a contradiction (and by lemma 1) suppose C({x,y}) = {y}. Then by axiom 1, C({x,z}) = {z} and C({y,z}) = {z}. By axiom 2, $z \notin$ C({x,y,z}). Suppose {y} = C({x,y,z}). Then by axiom 4, C({y,z}) = {y} contradicting C({y,z}) = {z}. Hence by lemma 2, {x} = C({x,y,z}). By axiom 3, there exists $a \in$ {y,z} such that $b \in$ C({a,b}) whenever $b \in \{x,y,z\} \setminus \{a\}$.Suppose a =y.Then $x \notin$ C({x,y}) = {y}. On the other hand if a = z, then $y \notin$ C({y,z}) = {z}. Hence there does not exist $a \in$ {y,z} such that $b \in$ C({a,b}) whenever $b \in \{x,y,z\} \setminus \{a\}$, contradicting axiom 3. This proves the claim.

Step 2 : Claim : Let $A \in [X]$ with #A > 2. Then C(A) = S(A).

Proof of Claim : Let {x} = C(A), where we are appealing to lemma 2 for the single value property of C(A). By axiom3,there exists a ∈ A \ {x} such that b ∈ C({a,b}) whenever b ∈ A \ {a}. By step 1, a > b whenever b ∈ A \ {a}. Thus, a > x. Towards a contradiction suppose that there exists b ∈ X \ {a, x} such that a > b > x. Then by step 1, {x} = C({a,x}) = C({b,x}). By axiom 2, x ∉C(A) contradicting {x} = C(A). Hence there does not exist b ∈ A \ {a, x} such that a > b > x. This combined with a > x, proves that {x} = S(A) and consequently the claim.

Combining the two claims we obtain a proof of the lemma.

Proof of Theorem 1 : Follows from lemma 3 and the observation that S satisfies axioms 1 to 4.

Proof of Theorem 2 : Follows from theorem 1, proposition 3 and the observation that S satisfies axioms 1,2',3' and 4.

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