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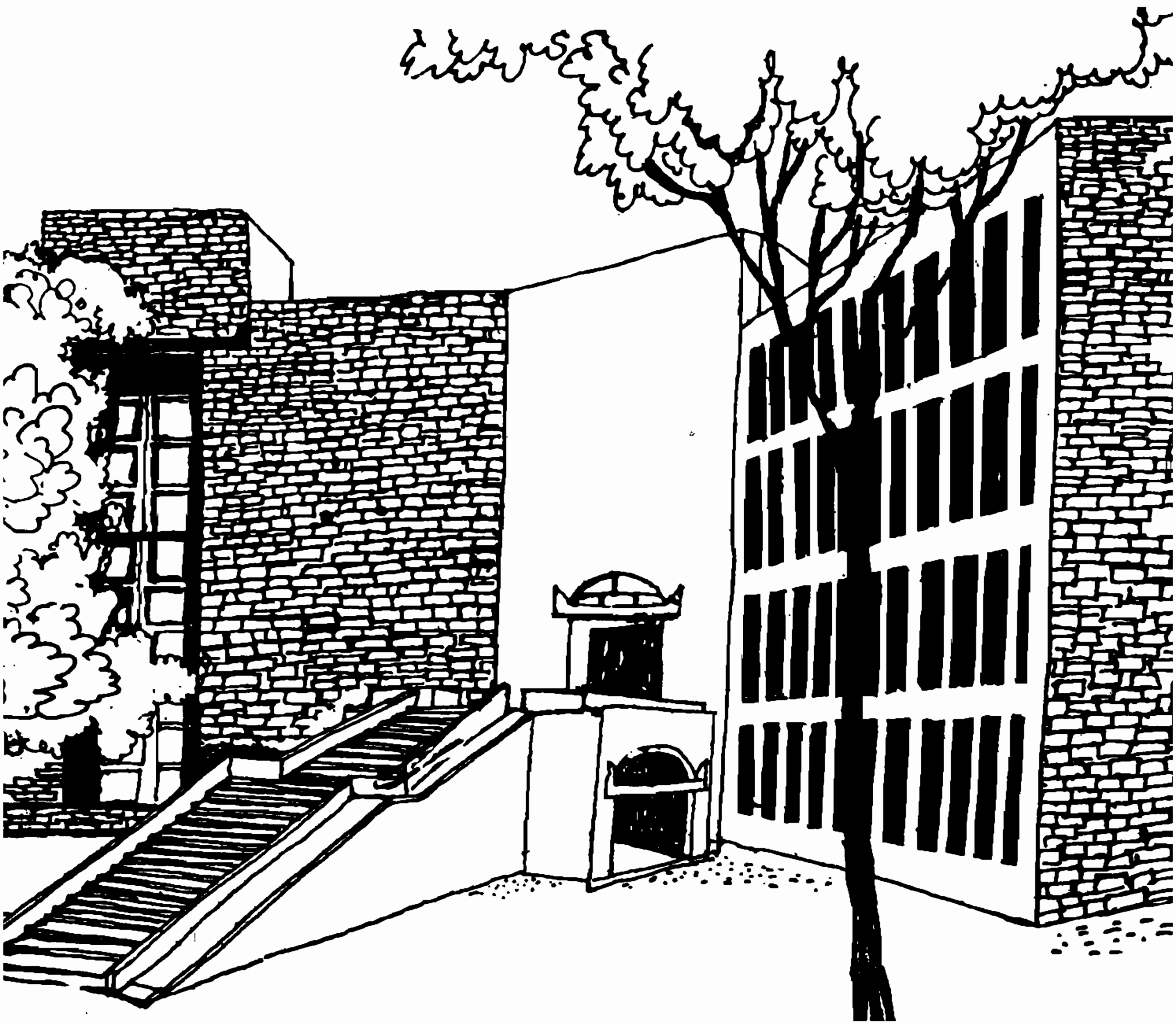

SOME SOLUTIONS FOR ABSTRACT GAMES: AXIOMATIC CHARACTERISATIONS

By

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<u>Abstract</u>

for

Some Solutions for Abstract Games : Axiomatic Characterizations

by

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In this paper we consider binary relations which are reflexive and complete.Such binary relations are referred to in the literature as abstract games.Given an abstract game a (game)solution is a function which associates to each subset a non-empty collection of points of the subset An important consequence of this framework is that often,a set may fail to have an element which is best with respect to the given binary relation.To circumvent this problem the concept of the top cycle set is introduced,which selects from among the feasible alternatives only those which are best with respect to the transitive closure of the given relation.The top cycle set is always non-empty and in this paper we provide an axiomatic characterization of the top-cycle solution.It is subsequently observed that the top cycle solution is the coarsest solution which satisfies two innocuos assumptions. In the final section of this paper we address the problem of axiomatically characterizing the uncovered solution (where 'covering' is now defined as a 'menu-based' concept).

Some Solutions for Abstract Games : Axiomatic Characterizations

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1. Introduction : An abiding problem in choice theory has been one of characterizing those choice functions which are obtained as a result of some kind of optimization. Specifically, the endeavour has concentrated largely on finding a binary relation (if there be any) whose best elements coincide with observed choices. An adequate survey of this line of research till the mid eighties is available in Moulin [1985]. More recently, the emphasis has focused on binary relations defined on non-empty subsets of a given set, such that the choice function coincides with the best subset corresponding to a feasible set of alternatives. This problem has been provided with a solution in Lahiri [1999], although the idea of binary relations defined on subsets is a concept which owes its analytical origins to Pattanaik and Xu [1990].
Given a binary relation, the idea of a function which associates with each set a non-empty subset of the given set has a long history whose exact origin is very difficult to specify and in any case is unknown to the author.In Laslier [1997] can be found a very exhaustive survey of the related theory when the given binary relation is reflexive,complete and anti-symmetric.
In this paper we extend the above set of binary relations to include those which are not necessarily anti-symmetric.Such binary relations which are reflexive and complete are referred to in the literature as abstract games.In a recent paper,Peris and Subiza [1999], refer to abstract games as weak tournaments.Given an abstract game, a (game)solution is a function which associates to each non-empty subset a non-empty collection of elements from the subset, on the basis of the given abstract game.Lucas [1992] has a discussion of abstract games and related solution concepts, particularly in the context of cooperative games.Much of what is discussed in Laslier [1997] and references therein carry through into this framework.An important consequence of both the frameworks is that often,a set may fail to have an element which is best with respect to the given binary relation.To circumvent this problem the concept of the top cycle set is introduced,which selects from among the feasible alternatives only those which are best with respect to the transitive closure of the given relation.The top cycle set is always non-empty and in this paper we provide an axiomatic characterization of the top-cycle solution.It is subsequently observed that the top cycle solution is the coarsest solution which satisfies two innocuos assumptions.
In Dutta and Laslier [1999] one finds the device of a comparison function, which is basically a real valued function defined on all pairs of alternatives satisfying the condition that the value of an ordered pair is negative of the value of the ordered

pair which is obtained by interchanging the order of the first ordered pair. Hence, in particular the value of the function along the diagonal (i.e. the set of ordered pairs with identical first and second components) is zero. A comparison function simultaneously captures the idea of preference and the intensity of preference. An alternative 'x' is preferred to another alternative 'y' if and only if the value of the comparison function at (x, y) is positive, and the value of the comparison function at (x, y) is meant to convey the intensity with which 'x' is preferred to 'y'. The device of a comparison function is a generalization of the concept of a binary relation.With the help of a comparison function they introduce the notion of 'cover': 'x' is said to cover 'y' if 'x' is preferred to 'y' (i.e. the value of the comparison function at (x, y) is positive) and for every other third element 'z' the value of the comparison function at the ordered pair (x, z) is alteast as much as the value of the comparison function at the ordered pair (y, z). Given any feasible set, its uncovered set is the set of all elements in the feasible set which are not covered by any other element in the same set. The question that naturally arises is the following : Given a choice function , under what condition does a comparison function exist, whose uncovered sets always coincide with the choice function? This question has been discussed in Lahiri [1999], where it is observed that the binary relation 'is uncovered' is reflexive, complete and quasi-transitive and any reflexive, complete and quasi-transitive binary relation can be made to coincide with the "is uncovered" relation of some comparison function. The problem becomes much more difficult if instead of defining the covering relation globally, we considered the covering relation for each individual feasible set, by simply looking at the restriction of the comparison function to that set. In such a situation that fact that 'x' covers 'y' in a particular feasible set does not imply that 'x' covers 'y' globally. In effect, we are then concerned with what Sen [1997] calls 'menu based' relations.

In the final section of this paper we address the problem of axiomatically characterizing the uncovered solution (where 'covering' is now defined as a 'menu-based' concept), by considering only those comparison functions which can assume only three values : 1, 0 and -1. These comparison functions are nothing but abstract games.

In Peris and Subiza [1999] it is shown that a considerable portion of the theory developed in an earlier paper by Dutta [1999],in the context of tournaments, carry through to abstract games as well.Our axiomatic characterizations are however different from the ones available in either Dutta and Laslier [1999] or Peris and Subiza [1999].

2. Game Solutions :- Let X be a finite, non-empty set and given any non empty subset A of X, let [A] denote the collection of all non-empty subsets of A. Thus in particular, [X] denotes the set of all non-empty subsets of X. If $A \in [X]$, then # (A) denotes the number of elements in A. A binary relation R on X is said to be (a) reflexive if $\forall x \in X : (x, x) \in R$; (b) complete if $\forall x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$;(c) transitive if $\forall x, y, z \in X$, $[(x, y) \in R \;\&\; (y, z) \in R$ implies $(x, z) \in R]$;(d) anti-symmetric if $[\forall x, y \in X, (x, y) \in R \;\&\; (y, x) \in R$ implies $x = y]$. Given a binary relation R on X and $A \in [X]$, let $R|A =$

$R \cap (A \times A)$. Let Π denote the set of all reflexive and complete binary relations. If $R \in \Pi$, then R is called an abstract game. Given a binary relation R, let $P(R) = \{(x, y) \in R / (y, x) \notin R\}$ and $I(R) = \{(x, y) \in R / (y, x) \in R\}$. P(R) is called the asymmetric part of R and I(R) is called the symmetric part of R. Given a binary relation R on X and $A \in [X]$, let $G(A,R) =\{ x \in A / \forall y \in A :(x,y) \in R\}$. Given $A \in [X]$, let $\Delta (A)$ denote the diagonal of A i.e. $\Delta (A)=\{(x,x)/x\in A\}$.

The following example shows that given $R \in \Pi$ and $A \in [X]$, G(A,R) may be empty:

Example 1:Let $X =\{x,y,z\}$ and let $R =\Delta (X)\cup\{(x,y),(y,z),(z,x)\}$.Clearly G(X,R) is empty.

Given $R \in \Pi$, $A \in [X]$,let $T(R|A)$ be a binary relation on A defined as follows: $(x, y) \in T(R|A)$ if and only if there exists a positive integer K and $x_1,\ldots,x_K$ in A with (i) $x_1 = x$, $x_K = y$: (ii) $(x_i, x_{i+1}) \in R$ $\forall$ $i \in \{1,\ldots K-1\}$. $T(R|A)$ is called the transitive hull of R in A.Clearly $T(R|A)$ is always transitive.

Given $R \in \Pi$, $A \in [X]$, $G(A,T(R|A))$ is called the top cycle set of R in A.Clearly $G(A,T(R|A))$ is non-empty whenever $R \in \Pi$ and $A \in [X]$.

Let, R belong to Π. An R-based game solution on X is a function $S: [X] \to [X]$ such that:

(i) $\forall A\in [X]:S(A) \subset A$;

(ii) $\forall x,y\in X :x\in S(\{x,y\})$ if and only if $(x,y) \in R$.

Thus in particular, $R = R^S \equiv \cup\{ S(\{x,y\}) \times \{x,y\}/x,y \in X \}$

If $\forall A\in [X]$, G(A,R) is non-empty valued then the associated game solution is called the R- based best solution on X.In future, whenever there is no scope for confusion, an R- based game solution will be simply referred to as a solution.

The top cycle solution denoted $TC: [X] \to [X]$ is defined as follows: $\forall A\in [X]$: $TC(A) = G(A,T(R|A))$.

Given $R \in \Pi$, $A \in [X]$ and $x, y \in X$, we say that x covers y via R in A if :

(i) $x, y \in A$; (ii) $(x, y) \in P(R)$; (iii) $\forall z \in A$: $[(y, z) \in R$ implies $(x, z) \in R]$; (iv) $\forall z \in A$: $[(y, z) \in P(R)$ implies $(x, z) \in P(R)]$.

Given $R \in \Pi$, let $\hat{R}(A) = \{(x, y) \in A\times A$ / x covers y via R in A$\}$. Let $UC(A) = \{x \in A$ / if $y \in A$ then $(y, x) \notin \hat{R}(A)\}$. It is easy to see that $\forall A \in [X]$, $\hat{R}(A)$ is a transitive binary relation on A. Thus $UC(A) \neq \phi$ whenever $A \in [X]$. Thus (i)$\forall A \in[X]$: $UC(A) \subset A$ $\forall A \in [X]$;(ii) $\forall x,y\in X:x\in UC(A)$ if and only if $(x,y) \in R$.

The solution $UC: [X]\to[X]$ is called the uncovered solution.

Given $A \in [X]$ and $x \in X$ let $s(x,A)=\#\{y\in A/(x,y) \in P(R)\} - \#\{y\in A/(y,x) \in P(R)\}$.

The Copeland solution $Co: [X]\to[X]$ is defined as follows:$\forall A \in[X]$:

$Co(A)=\{x\in A/\forall y\in A: s(x,A) \geq s(y,A)\}$.

The following proposition is available in Laslier [1997]:

Proposition 1: $\forall A \in[X]:Co(A)\subset UC(A)\subset TC(A)$.

Example 2:Let $X =\{x,y,z\}$ and let $R =\Delta (X)\cup\{(x,y),(y,z),(z,y),(x,z),(z,x)\}$.Now $Co(X)=\{x\}\subset\subset \{x,z\} =UC(X) \subset\subset X =TC(X)$.Further $Co(X) \subset\subset UC(X) \subset\subset TC(X)$.

3. Axioms for the Top Cycle Solution:A solution S on X is said to satisfy :
   Strong Condorcet (SC) if $\forall A \in[X]$: $[x\in A]$ and $[\forall y\in A\setminus\{x\}:(x,y) \in P(R)]$ implies $[S(A) =\{x\}]$;

Expansion Independence (EI) if $\forall$ A $\in$[X]:[x$\in$S(A),y$\in$A,(y,z) $\in$R] implies [x$\in$S(A$\cup${z})];
Existence of an Inessential Alternative (EIA) if $\forall$ A $\in$[X]$\times$ Π with #(A) $\geq$ 2 and $\forall$ x$\in$S(A), there exists y$\in$A (possibly depending on A and x) such that x$\in$S(A\{y}).
Theorem 1:The only solution on X which satisfies SC,EI and EIA is TC.
Proof:It is clear that TC satisfies SC,EI and EIA.Hence let S be any solution that satisfies SC,EI and EIA.Let A$\in$[X].If #(A) is one or two there is nothing to prove since S(A)=TC(A) by definition.Thus suppose S(A)=TC(A) whenever #(A)=1,...k.Let #(A)=k+1.Let x$\in$A.If $\forall$y$\in$A\{x}:(x,y) $\in$P(R)] then S(A) ={x}=TC(A).Hence suppose $\forall$x$\in$A there exists y $\in$A\{x} such that (y,x) $\in$R.
Let x $\in$TC(A).Since TC satisfies EIA,there exists z $\in$A such that x $\in$TC(A\{z}).By the induction hypothesis S(A\{z})= TC(A\{z}).If (x,z) $\in$R then by EI , x $\in$S(A).If (x,z)$\notin$R,then since x $\in$TC(A)= G(A,T(R|A)) there exists w$\in$A such that (x,w) $\in$T(R|A) and (w,z) $\in$R.Then by EI once again x $\in$S(A). Hence TC(A)$\subset$S(A).
Now suppose x $\in$S(A) and towards a contradiction suppose x $\notin$TC(A).By EIA there exists z $\in$A such that x $\in$S(A\{z}).By the induction hypothesis S(A\{z})= TC(A\{z}). If (x,z) $\in$R then by EI applied to TC, x $\in$TC(A). Hence suppose (x,z)$\notin$R.Thus (z,x)$\in$P(R). Let y $\in$TC(A). Clearly y$\neq$x.Suppose y$\neq$z.Thus y$\in$A\{z}.Thus (x,y)$\in$ T(R|A) which combined with y $\in$TC(A) gives us x $\in$TC(A).Hence y=z.If for some w $\in$A\{x,z} we had (w,z)$\in$R, then since x $\in$TC(A\{z}) and w $\in$A\{z} we would get x $\in$TC(A).Thus $\forall$w$\in$A:(z,w)$\in$P(R).But then by SC,S(A)={z}, contradicting x $\in$S(A).Thus x $\in$TC(A). Hence S(A)$\subset$TC(A). Thus S(A)=TC(A).
By a standard induction argument it now follows that $\forall$A$\in$[X]: S(A)=TC(A). ♣

A solution S on X is said to satisfy:
Converse Condorcet (CC) if $\forall$ A$\in$[X] and x$\in$A:[$\forall$y$\in$A\{x}:(y,x)$\in$P(R)] implies [x$\notin$S(A)];
Weak Existence of an Inessential Alternative (WEIA) if $\forall$ A $\in$[X] with #(A) $\geq$ 4 and $\forall$ x$\in$S(A), there exists y$\in$A (possibly depending on A and x) such that x$\in$S(A\{y}).
Since TC satisfies EIA it also saisfies WIEA. Infact we can now prove the following:
Theorem 2:Let S be any solution on X which satisfies SC,CC and WEIA.Then, $\forall$ A $\in$[X]: S(A) $\subset$ TC(A).
Proof: Step 1: Let S be any solution on X which satisfies SC and CC.Then, $\forall$ A $\in$[X] with #(A) $\leq$ 3: S(A) $\subset$ TC(A).
Proof of Step 1:For #(A) $\leq$ 2 there is nothing to prove since by the definition of a solution all of them agree on such sets.Hence suppose #(A) =3.Let A={x,y,z} with x$\neq$y$\neq$z$\neq$x.Suppose without loss of generality x$\in$S(A).If (x,y),(x,z) $\in$R,then x $\in$TC(A).Thus,suppose without loss of generality that (y,x) $\in$P(R).If (z,x) $\in$P(R) then by CC, x$\notin$S(A),contradicting what we have assumed.Hence (x,z) must belong to R.If (z,y) $\in$R,then again x $\in$TC(A).If (y,z) $\in$P(R),then by SC,S(A)={y},contradicting x$\in$S(A).Thus S(A) $\subset$ TC(A).
Step 2: Let S be any solution on X such that $\forall$ A $\in$[X] with #(A) $\leq$ 3: S(A) $\subset$ TC(A).Suppose S satisfies WEIA.Then, $\forall$ A $\in$[X]: S(A) $\subset$ TC(A).

Proof of Step 2: Suppose that $\forall A \in [X]$ with $3 \leq \#(A) \leq m : S(A) \subset TC(A)$.Let $\#(A) = m+1$.Thus $\#(A) \geq 4$.Let $x \in S(A)$.By WEIA,there exists $y \in A$ such that $x \in S(A \setminus \{y\})$.By the induction hypothesis $S(A\setminus\{y\}) \subset TC(A\setminus\{y\})$.Thus, $x \in TC(A\setminus\{y\})$.If $(x,y) \in R$, then clearly $x \in TC(A)$.Hence, $S(A) \subset TC(A)$.Suppose $(y,x) \in P(R)$.If $\forall z \in A\setminus\{y\}:(y,z) \in P(R)$,then by SC,$S(A)=\{y\}$,contradicting $x \in S(A)$.Hence ,there exists $z \in A\setminus\{x,y\}$ such $(z,y) \in R$.Since, $x \in TC(A\setminus\{y\})$ and $z \in A\setminus\{y\}$, $(z,y) \in R$ implies $x \in TC(A)$.Thus $S(A) \subset TC(A)$.
Step 2 combined with Step 1 and a standard induction argument proves the theorem. ♣
Infact,the above proof reveals the following:
Theorem 3:Let S be any solution on X which satisfies SC and EIA.Then, $\forall A \in [X]$: $S(A) \subset TC(A)$.
CC is not required once we replace WEIA by EIA,since then the induction argument can begin from $\#(A) \geq 2$.

4. The Uncovered Solution: A solution S on X is said to satisfy Expansion (E) if $\forall A,B \in [X]$: $S(A) \cap S(B) \subset S(A \cup B)$.
It is easy to see that both TC and UC satisfy E:
(i)Let $A,B \in [X] \times \Pi$ and suppose $x \in UC(A) \cap UC(B)$.Towards a contradiction suppose that $x \notin UC(A \cup B)$.Hence there exists $y \in A \cup B$,such that y covers x via R in $A \cup B$.Without loss of generality suppose $y \in A$.Since $x \in A$, y covers x via R in A.This contradicts $x \in UC(A)$.Thus UC satisfies E.
(ii) Let $A,B \in [X]$ and suppose $x \in TC(A) \cap TC(B)$.Towards a contradiction suppose that $x \notin TC(A \cup B)$.Hence there exists $y \in A \cup B$,such that $(x,y) \notin T(R|A \cup B)$. Without loss of generality suppose $y \in A$.Thus $(x,y) \notin T(R|A \cup B)$ implies that $(x,y) \notin T(R|A)$.This contradicts $x \in TC(A)$.
Moulin [1986] has established the following:
Proposition 2:Let S be any solution satisfying SC and E. If[$\forall A \in [X]$ with $\#(A)=3$ we have $UC(A) \subset S(A)$] then [$\forall A \in [X]$: $UC(A) \subset S(A)$](see Appendix for details).
A solution S on X is said to satisfy Contraction (Con) if $\forall A \in [X]$ with $\#(A) \geq 4$,[$x \in S(A)$] implies [there exists a positive integer $K \geq 2$ and sets $A_1,\ldots,A_K \in [A]\setminus\{A\}$ such that (i)$\cup\{A_k / k=1,\ldots K\} = A$;(ii) $x \in \cap\{S(A_k)/k=1,\ldots K\}$].
Dutta and Laslier [1999] establish that UC satisfies Con.However,TC does not as the following example reveals:
Example 3:Let $X = \{x,y,z,w\}$ where x,y,z,w are all distinct. Let,$R = \Delta(X) \cup \{(x,y),(z,x),(w,x),(y,z),(w,y),(z,w)\}$.Clearly,$x \in TC(X)$.Let $A \in [X]\setminus\{X\}$, with $\#(A) \geq 2$. Suppose,$y \notin A$.Then,$x \notin TC(A)$.Hence, $x \in TC(A)$ and $\#(A) \geq 2$ implies $y \in A$.Suppose $x,y \in A \cap B$ where $A,B \in [X]\setminus\{X\}$, $A \neq B$, $A \not\subset B \not\subset A$.Without loss of generality suppose $A=\{x,y,z\}$ and $B=\{x,y,w\}$.Then, $x \notin TC(B)$.Thus,TC does not satisfy Con.
A solution S on X is said to satisfy:
Tie Splitting (TS) if $\forall A, B \in [X] \times \Pi$ with $A \cap B = \phi$: [$A \times B \subset I(R)$ implies $S(A \cup B) = S(A) \cup S(B)$] ;

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Strong Type 1 Property (ST1P) if $\forall$ x, y, z $\in$ X; [(y,x) $\in$P(R), (x,z) $\in$P(R), (z,y) $\in$R] implies S({x, y, z}) = {x, y, z}.
Note: Let S be any solution satisfying E. If S satisfies ST1P then,[$\forall$ A $\in$[X] with #(A)=3 we have UC(A) $\subset$ S(A)].

Proposition 3 :- Let S be a solution on X such that S(A) = UC(A) $\forall$ A $\in$ [X]. Then S satisfies SC, CC, TS, ST1P, E and Con.
Proof : We have already seen that UC satisfies E, and SC,CC,TS,ST1P being easy to verify let us show that S satisfies Con. Let A $\in$ [X] with # (A) $\geq$ 4 and x $\in$ S(A). Thus, y $\in$ A, y $\neq$ x implies either [(x, y) $\in$ R] or [there exists z_y $\in$ A with either ((x, z_y) $\in$ R and (y, z_y) $\notin$ R) or ((x, z_y) $\in$ P(R) and (y, z_y) $\notin$ P(R))]. Let A_o = {y $\in$ A / (x, y) $\in$ R}. Clearly $A_o \neq \phi$, since x $\in$ A_o. Further, since there does not exist y $\in$ A_o, such that y covers x via R in A_o, x $\in$ S(A_o).
Case 1:- A_o = A. Since # (A) $\geq$ 4, there exists $\bar{y}$ $\in$ A \ {x} such that A \ {x, $\bar{y}$} $\neq \phi$. Let A_1 = {x, $\bar{y}$} and A_2 = A - {$\bar{y}$}. Clearly $A_1 \subset\subset$ A, $A_2 \subset\subset$ A and $A_1 \cup A_2$ = A. Further x $\in$ S(A_1) $\cap$ S(A_2).
Case 2 : $A_o \subset\subset$ A. In this case, let A_1 = A_o and for y $\in$ A \ A_1, let A_y = {x, y, z_y}. Since # (A) $\geq$ 4, $A_y \subset\subset$ A whenever y $\in$ A\$A_1$. Further, $\forall$ y $\in$ A\$A_1$:x $\in$ S(A_y). Also, $A_1 \cup \left(\bigcup_{y \in A \setminus A_1} A_y\right)$ = A. Hence S satisfies Con. ♣

Lemma 1 : If # (X) $\leq$ 3 and S is a solution on X which satisfies SC, TS and ST1P, then S is the uncovered solution.
Proof : Let S and X be as in the statement of the lemma. If # (X) = 1 or 2, there is nothing to prove since S(A) = UC(A) $\forall$ A $\in$ [X] by the definition of a solution. Hence suppose # (X) = 3. If A $\in$ [X] and # (A) $\leq$ 2, then S(A)= UC(A). Thus suppose A = X = {x, y, z} with x $\neq$ y $\neq$ z $\neq$ x. If there exists a $\in$ X : (a,b) $\in$P(R) $\forall$ b $\in$ X, then S(X) = {a} = UC(X), by SC of both S and UC. Hence suppose that $\forall$ a $\in$ X, there exists b $\in$ X\{a} :(b,a) $\in$R.
Case 1 : I(R) = X. Then by TS of C and UC, S(X) = UC(X) = X.
Thus without loss of generality suppose, (x, y) $\in$P(R). Hence, by what has been mentioned before Case 1, (z,x) $\in$R.
Case 2 : (z,x), (y,z) $\in$P(R).
By ST1P, S(X) = {x, y, z} = UC(X).
Case 3 : (z,x) $\in$P(R), (y,z) $\in$I(R).
By ST1P, S(X) = {x, y, z} = UC(X).
Case 4 : (z,x) $\in$I(R), (y,z) $\in$P(R).
By ST1P, S(X) = {x, y, z} = UC(X).
Case 5 : (z,x) $\in$I(R), (y,z) $\in$I(R).
Thus, {z} x {x, y}$\subset$ I(R).
By TS, S(X) = S({z}) $\cup$ S({x, y}) = {x, z} = UC(X).
This proves Lemma 1. ♣

A look at the proof of Lemma 1 reveals that we have essentially proved the following:

Lemma 2 : Let S be a solution on X which satisfies SC, TS and ST1P. Then $\forall A \in [X]$ with $\#(A) \leq 3$, S(A) = UC(A).

The above observation follows by noting that UC(A) depends on the restriction of R to A only.

Note : If in Lemma 1 (: or for that matter in Lemma 2), we replace SC by CC and E we do not get the desired result as the following example reveals :

Example 4: Let X = {x, y, z} with $x \neq y \neq z \neq x$. Let S(X) = {x, y}, where $R=\Delta(X)\cup\{(y,x),(y,z),(x,z)\}$.S satisfies CC, E, TS and ST1P, the last two properties being satisfied vacuously. However, UC(X) = {y} $\neq$ S(X). Note that S does not satisfy SC, since $(y,x),(y,z) \in P(R)$and yet $S(X) \neq \{y\}$.

In Dutta and Laslier [1999] we find the following property for a solution S on X:

Type One Property (T1P) : $\forall$ x, y, z $\in$ X: $[(y,x) \in P(R), (x,z) \in P(R), (z,x) \in I(R)]$ implies S({x, y, z}) = {x, y, z}.

Clearly T1P is weaker than ST1P. However, if we replace ST1P by T1P in Lemma 1 (: or Lemma 2), we do not get the desired result as the following example reveals.

Example 5: Let X = {x, y, z} with $x \neq y \neq z \neq x$. Let S(X) = {x}, where $R=\Delta(X)\cup\{(x,y),(y,z),(z,x)\}$.Clearly S satisfies SC, TS, E, CC and T1P (: all vacuously). However, S violates ST1P which under the present situation would require S(X) = X. Further S(X) $\neq$ UC(X) = X.

We are now equipped to prove the following theorem:

Theorem 4 : A solution S on X is the uncovered solution if and only if S satisfies SC, TS, ST1P, E and Con.

Proof : Proposition 3 tells us that an uncovered solution satisfies all the properties mentioned in the theorem. Hence let S be a solution on X satisfying SC, TS, ST1P and Con. Let $R \in \Pi$. By Lemma 2, S(A) = UC(A) $\forall A \in [X]$ with $\#(A) \leq 3$. Suppose S(A) = UC(A) $\forall A \in [X]$ with

$\#(A) = 1,\ldots,m$, and let $B \in [X]$ with $\#(B) = m+1$. Let $x \in S(B)$. Suppose $m+1 \geq 4$, for otherwise there is nothing to prove. Hence by Con there exists a positive integer K and non-empty proper subsets $B_1, \ldots, B_K$ such that $B = \bigcup_{i=1}^{K} B_i$ and $x \in \bigcap_{i=1}^{K} S(B_i, R)$.

Clearly $\#(B_i) \leq m$ whenever $i \in \{1, \ldots, K\}$.

By our induction hypothesis, $S(B_i) = UC(B_i)$ $\forall i \in \{1, \ldots, K\}$. Thus $x \in \bigcap_{i=1}^{K} UC(B_i)$, and by E, $x \in UC(B)$. Thus, $S(B) \subset UC(B, R)$. By an exactly similar argument with the roles of S an UC interchanged, we get $UC(B) \subset S(B)$. By a standard induction argument, the theorem is established. ♣

Note : The above theorem is not valid without E or Con.

Example 6: Let X = {x, y, z, w} where all of them are distinct. Let S(X) = {x}, S(A)=A if $\#(A) = 3$, where $R=\Delta(X)\cup\{(x,y),(y,z),(z,w),(w,x),(x,z),(z,x),(y,w),(w,y)\}$. S satisfies SC, ST1P, TS (vacuously). Further, let A_1= {x, y} and A_2 = {x, z, w}. $x \in S(X)$ and $x \in S(A_1) \cap S(A_2)$. Further $A_1 \cup A_2 = X$, with $A_1 \subset\subset X$ and $A_2 \subset\subset X$. Thus S satisfies

Con. However, UC(X) = X ≠ {x} = S(X). Observe that, S does not satisfy E ,since y ∈ S({x, y, z}) ∩ S({y, z, w}) but y ∉ S(X).

Example 7: Let X be as above. Let S(X) = {x, y}, S(A) = {x} if x ∈ A, S(A) = A if x ∉ A where R=Δ(X)∪({x}×X)∪({y,z,w}×{y,z,w}).Clearly S satisfies SC, ST1P (:vacuously), TS and E. But S does not satisfy Con : y ∈ S(X). If we take any finite number of non-empty proper subsets of X whose union is X, atleast one must contain 'x' and thus its choice set cannot contain 'y'.

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Appendix

Here we provide a concise proof of Proposition 1.

Proof of Proposition 1:Clearly the proposition holds for #(A) = 1 or 2. Hence assume that the proposition holds for #(A) = 1,...,K and now let #(A) = K+1. Suppose x ∈ Co(A) ∪UC(A).If, [∀y∈A\{x}:(x,y) ∈R] then, x∈ TC(A). Hence, suppose that there exists y∈A\{x} such that (y,x) ∈P(R).Clearly, x ∈ Co(A\{x}) ∪UC(A\{x}).By the induction hypothesis x ∈TC(A\{x},R). If [∀z∈A\{y}:(y,z)∈P(R)], then Co(A) ∪UC(A)= {y},contradicting x∈Co(A) ∪UC(A). Hence there exists z∈A\{y}:(z,y)∈R.Since z∈A\{y} and x ∈TC(A\{x}) clearly, (x,z) ∈T(R|A).Thus (x,y) ∈T(R|A).Thus,TC(A).The proposition now follows by induction on the cardinality of A.

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