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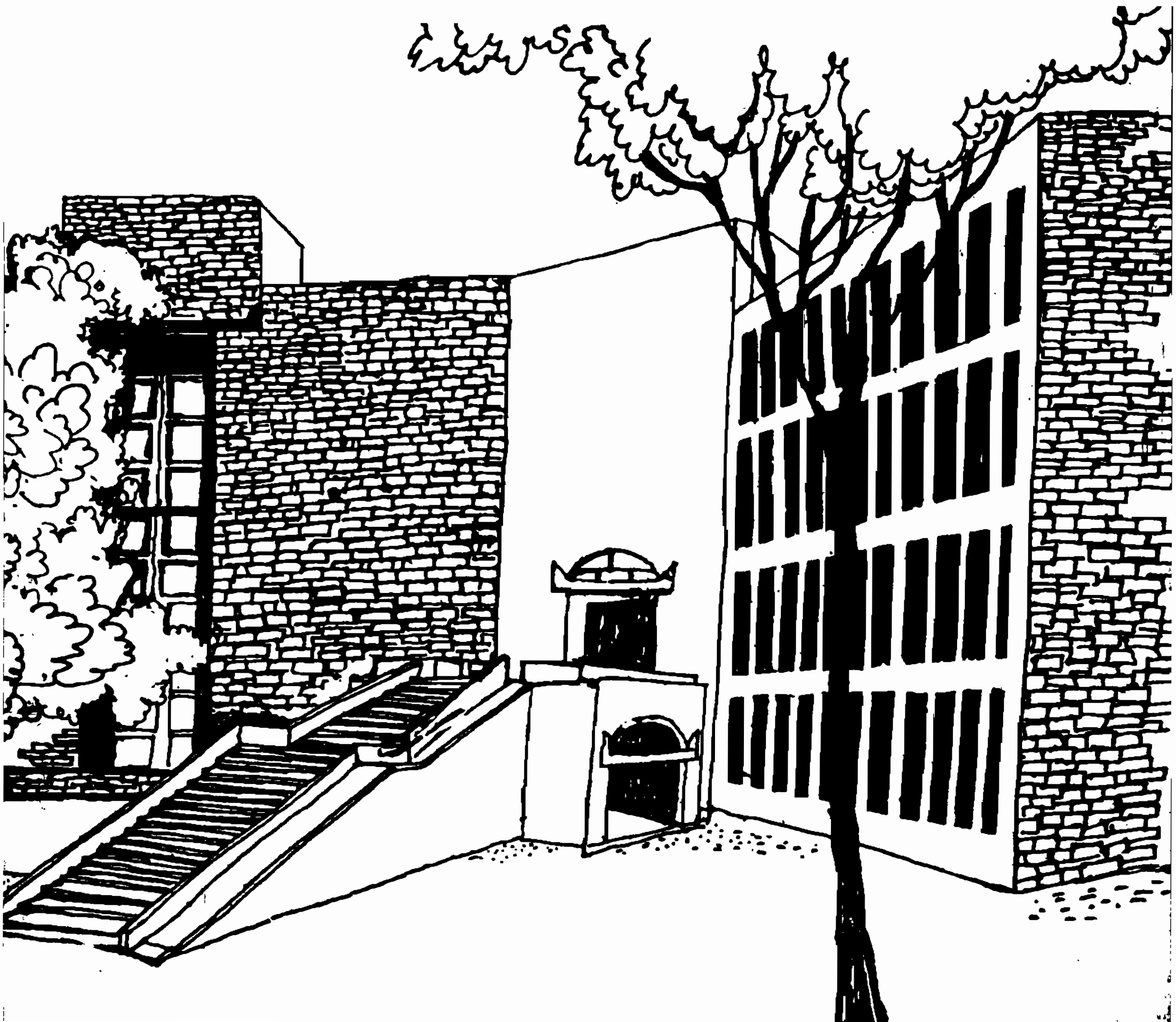

AXIOMATIC CHARACTERIZATION OF INDIRECT UTILITY AND LEXICOGRAPHIC EXTENSIONS

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

Axiomatic Characterization of Indirect Utility And Lexicographic Extensions

Somdeb Lahiri
Indian Institute of Management
Ahmedabad 380 015
India
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1. Introduction: The general problem we are interested in this paper is of the following variety: We are given a finite universal set and a linear ordering on it. What is the minimal axiomatic characterization of a particular extension of this linear ordering to the set of all non-empty subsets of the given set?

In Kannai and Peleg [1984] we find the starting point of this literature, which basically asserts that if the cardinality of the universal set is six or more, then there is no weak order on the power set which extends the linear order and satisfies two properties: one due to Gardenfors and the other known as Weak Independence. This result was followed by a quick succession of possibility results in Barbera, Barret and Pattanaik [1984], Barbera and Pattanaik [1984], Fishburn [1984], Heiner and Packard [1984], Holzman [1984], Nitzan and Pattanaik [1984] and Pattanaik and Peleg [1984]. Somewhat later, Bossert [1989] established a possibility result by dropping the completeness axiom for the binary relation on the power set and otherwise using the same axioms as in Kannai and Peleg [1984].

In recent times Malishevsky [1997] and Nehring and Puppe [1999] have addressed the problem of defining an "indirect utility preference". Malishevsky [1997] addresses the integrability problem: given a weak order on the power set, under what conditions is it an indirect utility preference? A similar question is also addressed in Nehring and Puppe [1999].

In our framework, a binary relation on the power set is an indirect utility extension if given two non-empty sets, the first is as good as the second if the best element (: with respect to the linear order) of the first set is as good as the best element of the second. In this paper, we provide a minimal set of assumptions which uniquely characterizes the indirect utility extension. The indirect utility extension is easily observed to be a slight modification of the weak ordering extension due to Barbera and Pattanaik [1984].

In a final section of this paper we consider the problem of axiomatically characterizing the so called "lexicographic" extension. It is similar to the extension considered by Bossert [1989]. However unlike the extension due to Bossert our extension is complete,and though it satisfies Gardenfor's Property it fails to satisfy Weak Independence. Given a set we consider the pair consisting of its best and worst point.Now given two sets the first is atleast as good as the second,if either the best point of the first set is better than the best point of the second or they both share the same best point,in which case the worst point of the first is required to be atleast as good as the worst point of the second. In a way,the decision maker becomes pessimistic only if he/she has not much to choose between the best points of two sets.

Similar results can be found in Pattanaik and Xu [1990] and Puppe [1996].

2.The Model: Let X be a given finite non-empty set and let [X] denote the set of all non-empty subsets of X. Let R be a given linear ordering on X i.e. R satisfies (i) reflexivity: $\forall x \in X, (x, x) \in R$; (ii) completeness: $\forall x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$; (iii) transitivity: $\forall x, y, z \in X$, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$; and (iv) antisymmetry: $\forall x, y \in X$, $[(x, y) \in R \;\&\; (y, x) \in R]$ implies $x = y$.

Let $\Re$ be a binary relation on [X]. It is said to be (i) reflexive if $\forall A \in [X], (A, A) \in \Re$;

(ii) complete if $\forall A, B \in [X]$ with $A \neq B$, either $(A, B) \in \Re$ or $(B, A) \in \Re$;

(iv) transitive if $\forall A, B, C \in [X]$, $[(A, B) \in \Re, (B, C) \in \Re]$ implies $(A, C) \in \Re$.

Let $I(\Re) = \{(A, B) \in \Re / (B, A) \in \Re\}$ and $P(\Re) = \{(A, B) \in \Re / (B, A) \notin \Re\}$.

Given $A \in [X]$, let g(A) be the unique element of A satisfying $(g(A), x) \in R$ whenever

$x \in A$ and let $l(A)$ be the unique element of A satisfying $(x, l(A)) \in R$ whenever $x \in A$.

A binary relation $\Re$ on [X] is said to satisfy

(a) Gardenfor's Property(GP) if $\forall A \in [X]$ and $x \in X \setminus A$, (i) $(x, g(A)) \in R$ implies $(A \cup \{x\}, A) \in P(\Re)$; (ii) $(l(A), x) \in R$ implies $(A, A \cup \{x\}) \in P(\Re)$.

(b) Weak Independence(W.IND) if $\forall A, B \in [X]$ with $(A, B) \in P(\Re)$, if $x \in X \setminus (A \cup B)$ then $(A \cup \{x\}, B \cup \{x\}) \in \Re$.

Kannai and Peleg [1984] show the following:

Theorem 1:- If X has atleast six elements, then there does not exist any binary relation on [X] which satisfies reflexivity, completeness, transitivity, GP and W.IND.

Bossert [1989] proves the existence of a unique binary relation on [X] which satisfies all the properties in Theorem 1 other than completeness.

Let $\overline{\Re} = \{(A,B) \in [X] \times [X] / (g(A), g(B)) \in R\}$. $\overline{\Re}$ is called the indirect utility extension. It is easy to see that $\overline{\Re}$ satisfies reflexivity, completeness, transitivity and W.IND, but does not satisfy GP. However, it satisfies the following property which modifies a similar one due to Barbera [1977]:

Property 1: $\forall x, y \in X, x \neq y, (x, y) \in R$ implies $(\{x\}, \{x, y\}) \in \Re$ and $(\{x, y\}, \{y\}) \in P(\Re)$.

Further $\overline{\Re}$ satisfies the following modification of W.IND:

Property 2: $(A, B) \in \Re$ and $x \in X \setminus (A \cup B)$ implies $(A \cup \{x\}, B \cup \{x\}) \in \Re$.

Note: Property 2 implies W.IND.

A property found in Nehring and Puppe [1999] is the following:

Monotonicity (MON): $\forall A, B \in [X], B \subset A$ implies $(A, B) \in \Re$.

It is not difficult to see that $\overline{\Re}$ satisfies (MON).

3. Axiomatic Characterization of the Indirect Utility Extension:-

<u>Theorem 2:</u> The only transitive binary relation on [X] to satisfy Property 1, Property 2 and MON is $\overline{\mathfrak{R}}$.

<u>Proof</u>: We have already seen that $\overline{\mathfrak{R}}$ satisfies the above mentioned properties. Hence let $\mathfrak{R}$ be transitive and satisfy Property 1, Property 2 and MON. By MON, $\mathfrak{R}$ must be reflexive. Let $A \in [X]$. Suppose $A = \{x_1, \ldots, x_n\}$ where $(x_i, x_{i+1}) \in R$ for $i \in \{1, \ldots, n-1\}$. If $n = 1$, then $A = \{x_1\} = \{g(A)\}$ and hence $(A, \{g(A)\}) \in I(\mathfrak{R})$. Hence suppose $n \geq 2$ and $2 \leq k \leq n$. Observe, $g(A) = x_1$ and $(x_1, x_2) \in R$ implies by Property 1, $(\{x_1\}, \{x_1, x_2\}) \in \mathfrak{R}$. Suppose $(\{x_1\}, \{x_1, \ldots, x_{k-1}\}) \in \mathfrak{R}$. Now $(x_{k-1}, x_k) \in R$ implies (by Property 1) that $(\{x_{k-1}\}, \{x_{k-1}, x_k\}) \in \mathfrak{R}$. By repeated application of Property 2, we get $(\{x_1, \ldots, x_{k-1}\}, \{x_1, \ldots, x_k\}) \in \mathfrak{R}$. Hence by transitivity of $\mathfrak{R}$, we get, $(\{x_1\}, \{x_1, \ldots, x_k\}) \in \mathfrak{R}$. We have seen that $(\{x_1\}, \{x_1, \ldots, x_k\}) \in \mathfrak{R}$ for k=1 and 2. Further since $(\{x_1\}, \{x_1, \ldots, x_{k-1}\}) \in \mathfrak{R}$ implies $(\{x_1\}, \{x_1, \ldots, x_k\}) \in \mathfrak{R}$, we have by finite mathematical induction that $(\{x_1\}, A) \in \mathfrak{R}$. By (MON), $(A, \{x_1\}) \in \mathfrak{R}$. Thus, $(A, \{g(A)\}) \in I(\mathfrak{R})$. Let $(A, B) \in [X]$. If $g(A) = g(B)$, then $(A, \{g(A)\}) \in I(\mathfrak{R})$ and $(\{g(B)\}, B) \in I(\mathfrak{R})$ implies $(A, B) \in I(\mathfrak{R}) \subset \mathfrak{R}$ (by transitivity of $\mathfrak{R}$). Hence suppose, $g(A) \neq g(B)$. Now, $(g(A), g(B)) \in \mathfrak{R}$ implies by Property 1 and transitivity of $\mathfrak{R}$, that $(\{g(A)\}, \{g(B)\}) \in P(\mathfrak{R})$. Thus $(A, B) \in P(\mathfrak{R}) \subset \mathfrak{R}$, by transitivity of $\mathfrak{R}$, since $(A, \{g(A)\}) \in I(\mathfrak{R})$ & $(\{g(B)\}, B) \in I(\mathfrak{R})$.

Conversely suppose, $(A, B) \in \mathfrak{R}$. Towards a contradiction suppose, $(g(A), g(B)) \notin R$. By reflexivity and completeness of R, $g(A) \neq g(B)$ and $(g(B), g(A)) \in R$. By Property 1 and transitivity of $\mathfrak{R}$, $(\{g(B)\}, \{g(A)\}) \in P(\mathfrak{R})$. This combined with transitivity of $\mathfrak{R}$ and $(A, \{g(A)\})$, $(\{g(B)\}, B) \in I(\mathfrak{R})$ gives, $(B, A) \in P(\mathfrak{R})$, contradicting $(A, B) \in \mathfrak{R}$. Hence $(A, B) \in \mathfrak{R} \leftrightarrow (g(A), g(B) \in R$. Completeness of $\mathfrak{R}$ now follows from reflexivity and completeness of R.

Thus $\mathfrak{R} = \overline{\mathfrak{R}}$. Q.E.D.

<u>Logical Independence of Property 1, Property 2 and MON</u>:-Let X = {x, y, z} with $x \neq y \neq z \neq x$. Let R = {(x, y), (y, z), (x, z), (x, x), (y, y), (z, z)}.Given $A \subset [X]x\ [X]$,let T(A) denote the transitive hull of A.

<u>Example 1</u>:- Let $\Re$ = [X] x [X]. $\Re$ satisfies Property 2 and MON, but not Property 1, since $x \neq y$, $(x, y) \in R$ and yet $(\{x, y\}, \{y\}) \notin P(\Re)$.

<u>Example 2</u>:- $\Re = T(\{(\{x, y\}, \{y\}), (\{x, z\}, \{z\}), (\{ y, z\}, \{z\}), (\{x\}, \{x, y\}), (\{y\}, \{y, z\}),(\{x\},\{x,z\})\}) \cup \{(A, B) \in [X] \times [X] / B \subset A\}$.

Now $\Re$ satisfies MON and Property 1. However, $(\{x\}, \{x,y\}) \in \Re$, $z \notin \{x, y\}$ and yet $(\{x, z\}, \{x, y, z\}) \notin \Re$. Thus $\Re$ does not satisfy Property 2.

<u>Example 3</u>:- $\Re = T(\{(\{x, y\}, \{y\}), (\{x, z\}, \{z\}), (\{ y, z\}, \{z\}), (\{x\}, \{x, y\}),$
$(\{x\}, \{x, z\}),(\{y\},\{y, z\}), (\{x, y, z\}, \{y, z\}), (\{x,y, z\}, \{x, z\}),(\{x,z\},\{x,y,z,\}),$
$(\{x, y\}, \{x, y, z\})\})$. Here $\Re$ satisifies Properties 1 and 2. However, $\{x\} \subset \{x, y\}$ and $(\{x, y\}, \{x\}) \notin \Re$. Hence $\Re$ does not satisfy MON.

<u>Example 4</u>:- Let $\Re = T(\{(\{x\}, \{x, y\}), (\{y\}, \{y, z\}), (\{x\}, \{x, z\}), (\{x,y\}, \{y\}), (\{x, z\}, \{z\}), (\{y, z\}, \{z\})\})$. $\Re$ satisifies Property 1. However $y \notin \{x, z\}$, $(\{x\}, \{x, z\}) \in \Re$ and yet $(\{x, y\}, \{x, y, z\}) \notin \Re$. Thus $\Re$ does not satisfy Property 2. Since $(\{x, y\}, \{x\}) \notin \Re$ it does not satisfy MON either.

<u>Example 5</u>: Let $\Re = \{ (\{ x \} , \{ x, y \}) , (\{ x, z \} , \{ x, y, z, \}) \}$. $\Re$ does not satisfy Property 1, because, $(y, z,) \in R$, $y \neq z$ and yet $(\{ y, z \},y) \notin \Re$. Neither does, $(\{ y, z, \} , \{ z \})$ belong to $\Re$. However, $\Re$ satifies Property 2. Since $(\{ x, y \}, \{x \})\notin \Re$, $\Re$ does not satisfy MON.

<u>Example 6</u>: Let $\Re = T(\{(A, B) \in [X] \times [X] / B \subset A \} \cup \{ (\{x\},\{x,z\}) \})$. $\Re$ satisfies MON. However, $x \neq y$, $(x,y) \in R$ and yet $(\{x,y\},\{y\}) \})\notin \Re$.Thus $\Re$ does not satisfy Property

1.Further $y \notin \{x,z\}$, $(\{x\},\{x,z\}) \in \Re$ and yet $(\{x,y\},\{x,y,z\}) \notin \Re$. Thus $\Re$ doe

Property2.

Example 7 : Let $\Re = \{(A, B) \in [X] \times [X] / x \notin A \cup B\}$. $\Re$ does not satisfy Prc

$x \neq y$,$(x,y) \in R$ and yet $(\{x\},\{x,y\}) \notin \Re$. It does not satisfy Property 2, because,

$(\{y\},\{y,z\}) \in \Re$,$x \notin \{y,z\}$ and yet $(\{x,y\},\{x,y,z\}) \notin \Re$. It does not satisfy MON beca

$(\{x,y\},\{y\}) \notin \Re$.

The Lexicographic Extension :- A binary relation on [X] denoted $\Re^*$ is called the l

extension if $\Re^* = \{(A, B) \in [X] \times [X]$/either $(g(A), g(B)) \in P(R)$ or $[g(A) = g(B)$ an

$\in R]\}$.

It is easy to see that $\Re^*$ is reflexive, complete, transitive and satisfies GP.

neither satisfies W.IND or MON.

Example 8:- Let $X = \{x, y, z, w\}$ with all its elements distinct and suppose R is a li

X with $\{(x, y), (y, z), (z, w)\} \subset P(R)$. Let $A = \{y, w\}$ and $B = \{z\}$. Clearly (A, I

Further $x \notin A \cup B$ and yet $(B \cup \{x\}, A \cup \{x\}) \in P(\Re^*)$ contradicting W.IND.

$\Re^*$ satisfy MON, since $B \cup \{x\} \subset \{x, y, z, w\}$ and yet $(\{x, y, z, w\}, B \cup \{x\}) \notin \Re^*$

However, $\Re^*$ is not the only binary relation on [X] to satisfy GP.

Let $\Re_* = \{(A, B) \in [X] \times [X]$/either $(l(A), l(B) \in P(R)$ or $[l(A) = l(B)$ and $(g(A), g(E$

$\Re_*$ may be called the inverse lexicographic extension.

Example 9:- Let $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$ and let R be a linear order on X

$(y, z)\} \subset P(R)$. Now $(\{x, z\}, \{y\}) \in P(\Re^*) \setminus \Re_*$ and $(\{y\}, \{x, z\}) \in P(\Re_*) \setminus \Re^*$. Tl

However, $\Re_*$ satisfies GP.

To narrow down on $\Re^*$ we invoke the following two properties:

Property 3:- $\forall A \in [X]$ and $x, y \in X$,

(i) $(l(A),y) \in R$ implies $(A \cup \{x\}, \{y, x\}) \in \mathfrak{R}$

(ii) $(\{y\}, A) \in P(\mathfrak{R})$ implies $(\{y, x\}, A \cup \{x\}) \in \mathfrak{R}$.

Property 4:- $\forall x, y, z \in X$ with $\{(x, y), (y, z)\} \subset P(R)$, $(\{x, z\}, \{y\}) \in P(\mathfrak{R})$.

Note:- $\mathfrak{R}_*$ satisfies Property 3 but not Property 4. $\mathfrak{R}^*$ satisfies both Properties 3 and 4. $\mathfrak{R}_*$ satisfies the following property which $\mathfrak{R}^*$ does not:

Property 5:- $\forall x, y, z, \in X$ with $\{(x, y), (y, z)\} \subset P(R)$, $(\{y\}, \{x, z\}) \in P(\mathfrak{R})$.

Lemma 1:- Let $\mathfrak{R}$ satisfy transitivity, GP and Property 3. Then $\forall A \in [X]$, $(A, \{l(A), g(A)\}) \in I(\mathfrak{R})$.

Proof:- Let $A = \{x_1, \ldots, x_n\}$ where $(x_i, x_{i+1}) \in P(R)$ $\forall i \in \{1, \ldots, n-1\}$. By multiple applications of GP and transitivity we get $(\{x_1\}, \{x_1, \ldots, x_{n-1}\}) \in P(\mathfrak{R})$ and $(\{x_2, \ldots, x_n\}, \{x_n\}) \in P(\mathfrak{R})$.

By Property 3, $(\{x_1, \ldots, x_n\}, A) \in \mathfrak{R}$ and $(A, \{x_1, x_n\}) \in \mathfrak{R}$. Hence the lemma.

Q.E.D.

Lemma 2:- Let $\mathfrak{R}$ satisfy transitivity, GP and Property 3. Then $\forall x, y, z \in X$ with $\{(x, y), (y, z)\} \subset P(R)$, $(\{x,y\}, \{x,z\}) \in P(\mathfrak{R})$ and $(\{x,z\}, \{y,z\}) \in P(\mathfrak{R})$.

Proof:- By Lemma 1, $(\{x, y, z\}, \{x, z\}) \in I(\mathfrak{R})$ and by GP, $(\{x, y\}, \{x,y,z\}) \in P(\mathfrak{R})$. By transitivity, $(\{x, y\}, \{x, z\}) \in P(\mathfrak{R})$. Further by GP, $(\{x, y, z\}, \{y, z\}) \in P(\mathfrak{R})$. By transitivity, $(\{x, z\}, \{y, z\}) \in P(\mathfrak{R})$.

Q.E.D.

Theorem 3: Let $\mathfrak{R}$ be a binary relation on [X] which is reflexive, complete and transitive. Then

(i) $\mathfrak{R} = \mathfrak{R}^*$ if and only if $\mathfrak{R}$ satisfies GP, Property 3 and Property 4;

(ii) $\mathfrak{R} = \mathfrak{R}_*$ if and only if $\mathfrak{R}$ satisfies GP, Property 3 and Property 5.

Proof:- It is easy to see that $\mathfrak{R}^*$ and $\mathfrak{R}_*$ satisfy the desired properties respectively.

(i) Let us suppose that $\Re$ is reflexive, complete, transitive and satisfies GP, Property 3 and Property 4. Suppose $(A, B) \in I(\Re^*)$. Thus $g(A) = g(B)$ and $l(A) = l(B)$. By Lemma 1, $(A, \{g(A), l(A)\}) \in I(\Re)$ and $(B, \{g(B), l(B)\}) \in I(\Re)$. Hence $(A, B) \in I(\Re)$ by transitivity of $\Re$. Thus $I(\Re^*) \subset I(\Re)$.

Now suppose $(A, B) \in P(\Re^*)$.

<u>Case 1</u>: $g(A) = g(B)$. Thus $(l(A), l(B) \in P(R)$. By Lemma 2, $(\{g(A), l(A)\}, \{g(B), l(B)\}) \in P(\Re)$. By Lemma 1 and transitivity of $\Re$, $(A, B) \in P(\Re)$.

<u>Case 2</u>:- $(g(A), g(B)) \in P(R)$.

Suppose $(g(B), l(A)) \in P(R)$.

Then by Property 4, $(\{g(A), l(A)\}, \{g(B)\}) \in P(\Re)$. By GP and reflexivity, $(\{g(B)\}, \{g(B), l(B)\}) \in \Re$. Thus $(\{g(A), l(A)\}, \{g(B), l(B)\}) \in P(\Re)$. By Lemma 1 and transitivity of $\Re$, $(A, B) \in P(\Re)$.

Now suppose $l(A) = g(B)$.

By GP, $(\{g(A), l(A)\}, \{g(B)\}) \in P(\Re)$. By GP and reflexivity, $(\{g(B)\}, \{g(B), l(B)\}) \in \Re$. By transitivity, $(\{g(A), l(A)\}, \{g(B), l(B)\}) \in P(\Re)$. By Lemma 1 and transitivity of $\Re$, $(A, B) \in P(\Re)$. Now suppose $(l(A), g(B)) \in P(R)$. By GP and reflexivity, $(\{g(A), l(A)\}, \{l(A)\}) \in \Re$. By GP, $(\{l(A)\}, \{l(A), g(B)\}) \in P(\Re)$ and $(\{l(A)\}, g(B)\}, \{g(B)\}) \in P(\Re)$. By transitivity $(\{l(A)\}, \{g(B)\}) \in P(\Re)$. By GP and reflexivity, $(\{g(B)\}, \{g(B), l(B)\}) \in P(\Re)$. By Lemma 1 and transitivity, $(A, B) \in P(\Re)$. Thus $P(\Re^*) \subset P(\Re)$.

Now $I(\Re^*) \cap P(\Re^*) = I(\Re) \cap P(\Re) = \phi$ and $I(\Re^*) \cup P(\Re^*) = I(\Re) \cup P(\Re) = [X] \times [X]$ by completeness.

$\therefore I(\Re) = I(\Re^*)$ and $P(\Re^*) = P(\Re)$

$\therefore \mathfrak{R}^* = \mathfrak{R}$.

(ii) The proof that $\mathfrak{R} = \mathfrak{R}_*$ if $\mathfrak{R}$ satisfies reflexivity, completeness, transitivity, GP, Property 3 and Property 5 is similar.

Q.E.D.

We have seen that $\mathfrak{R}_*$ ($\neq \mathfrak{R}^*$) satisfies GP, Property 3 and not Property 4.

$\overline{\mathfrak{R}}$ satisfies Property 3 and Property 4 but not GP.

Let $\mathfrak{R}_1 = \{(A, B) \in [X] \times [X] / (g(A), (g(B)) \in \mathfrak{R}$ & $(l(A), l(B)) \in R\}$ and $\mathfrak{R}_2 = \{(A, B) \in [X] \times [X] / (A, B) \notin \mathfrak{R}_1$ & (mid (A), mid (B)) $\in R\,)\}$ where if $A = \{x_1, \ldots, x_n\}$ with $(x_{1,} x_{i+1}) \in P(R)\ \forall\ i \in \{1, \ldots, n-1\}$, mid (A) $= x_{(n+1)/2}$ if n is odd

$= x_{n/2}$ if n is even.

Here 'mid' is the short form for middle.

Let $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$. $\mathfrak{R}$ satisfies GP and Property 4. However, if $X = \{x, y\ z, w\}$with $\{\ (x, Y), (y, z)\}\ (z, w)\} \subset P(R)$ where $\mathfrak{R}$ is a linear order, then if $A = \{x, z, w\}$ then $(\{y\}, A) \in P(\mathfrak{R})$. However, $(\{y, w\}, A \cup \{w\}) = (\{y, w\}, A) \notin \mathfrak{R}$. In fact $(A, \{y, w\}) \in P\ (\mathfrak{R})$. Thus $\mathfrak{R}$ does not satisfy Property 3.

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