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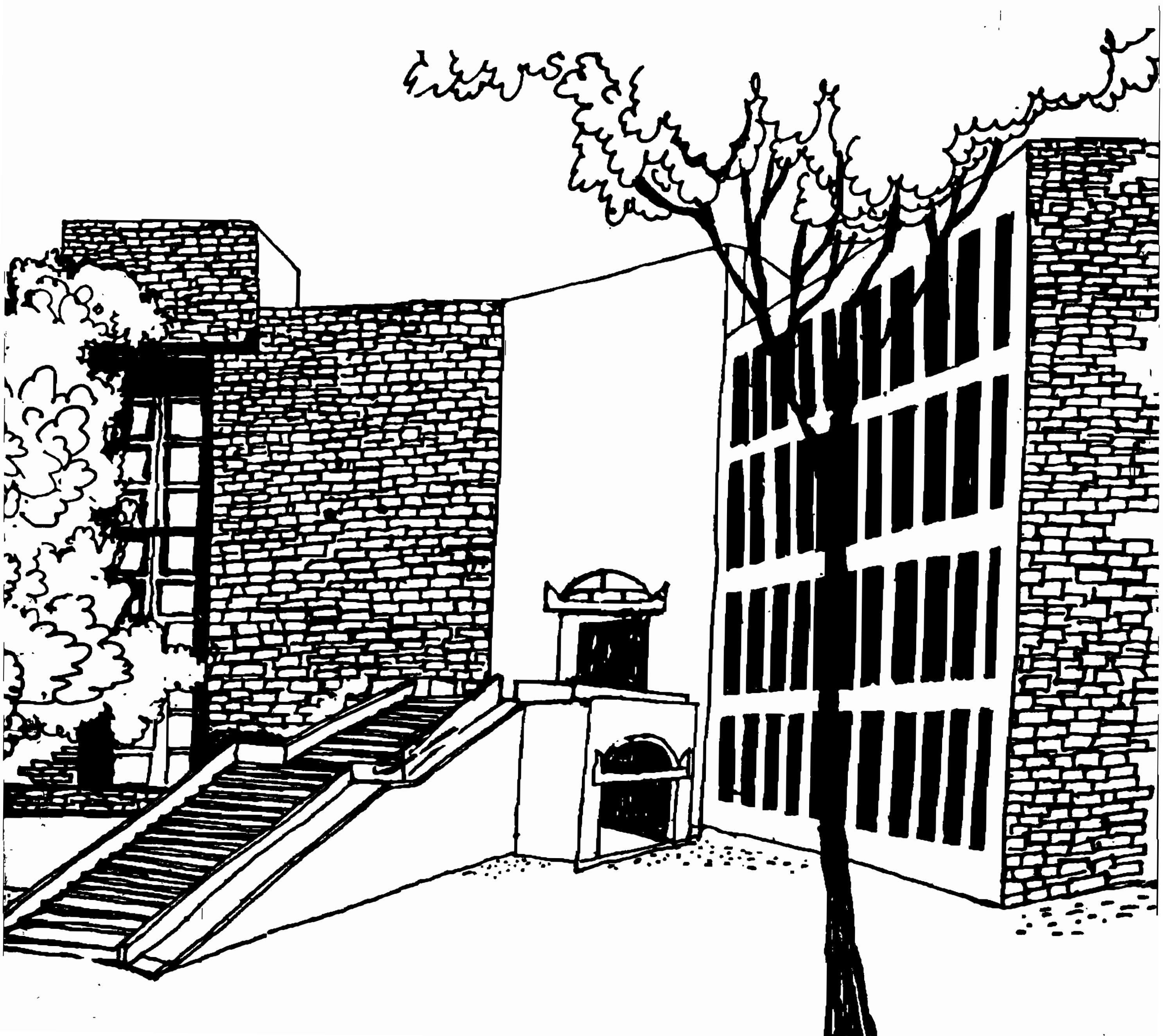

THE ONE DIMENSIONAL KAKUTANI'S FIXED POINT THEOREM: A CLASSROOM CAPSULE

By

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Introduction :- In problems of fair division of a given bundle of infinitely divisible commodities amongst a finite number of agents, the standard framework pioneered by Thomson [1988] has been one where a choice correspondence associates with each profile of preferences and a given aggregate initial endowment vector , a subset of the set of feasible allocations. The literature on this topic is so vast that the possibility of a single survey doing justice to all aspects of the problem is rather remote. However, a near adequate survey of the relevant literature is the monograph by Thomson [1995].

A feature of the study of such problems which is difficult to miss is that there is a clear dichotomy between the analytical methodology concerning the study of problems of fair division of just one commodity and the analytical techniques involving the study of problems of fair division of more than one commodity. However, within the study of problems concerning the fair division of more than one commodity, there does not appear to be a major difference involving the number of commodities. This observation by and large applies to the theorems, examples and counter examples pertaining to the relevant literature. To an extent, this phenomenon is not very surprising. The major difference that arises between one commodity fair division problems and multi-commodity fair division problems is the presence of the possibility of trading off the consumption of one commodity for another in the latter case and its absence in the former. This possibility, to the extent that it is invoked in the analysis of fair division problems does not depend on the number of commodities involved provided, the number of commodities is atleast two.

In this scenario, a particular choice correspondence which plays a significant role is the Walrasian choice correspondence particularly the one which results from equal division. Often it is necessary to assume the existence of a Walrasian equilibrium for each realisation of the economic environment i.e. profile of preferences and initial endowment. If the profile of preferences are strictly convex(i.e. the strict convex combination of two equally desirable commodity bundles is strictly preferred to each of them), then the proof of the existence of a Walrasian equilibrium follows as a simple consequence of a fixed point theorem which in itself is a corollary of the intermediate value theorem for continuous functions. However, the received theory of problems of fair division has a slightly wider scope, allowing for preferences which are not necessarily strictly convex, but just convex(i.e. the strict convex combination of two equally desirable commodity bundles is no worse than each of them).In such a situation the economy does not give rise to a continuous demand function to which the fixed point consequence of the intermediate value theorem can be applied. We are compelled to appeal to the general multidimensional version of Kakutani's fixed point theorem, which one usually obtains by applying a fixed point theorem for continuous function and an appropriate selection theorem [see Hildenbrand and Kirman (1988)]. The selection theorem is far from trivial and is almost as labor intensive as the fixed point theorem.

In this paper, we provide a simple proof of the one-dimensional version of the Kakutani's fixed point theorem, which is required to prove the existence of a Walrasian equilibrium in a two commodity multi-agent economy where preferences are convex. In a final section of this paper we apply the Kakutani's fixed point theorem to prove the existence of what we call a constrained equilibrium with rationing (and with fixed prices) in a model where each individual is endowed with positive quantities of two goods.The solution is very similar to the solution due to Dreze (1975).It is a problem of resource allocation from an initial position and the solution we propose falls within the general category of non-Walrasian equilibria.

The purpose of this paper is not to restrict the scope of general equilibrium theory. It simply cannot be, given that some of the simplest intertemporal general equilibrium models require an infinite dimensional commodity space. Nor is the purpose of this paper to restrict the scope of the study of problems of fair division. The fact that in this framework, no major result or axiom is sensitive to the number of commodities, does not mean that future ramifications of this literature will not involve such possibilities. The purpose of this paper is largely pedagogical.

The framework for the study of fair division of a bundle of infinitely divisible commodities amongst a finite number of agents that we have outlined above, is not all that matters in economic theory. However, it is one among those which have proved itself equal to the task of conveying some very deep consequences in social choice theory. Thus an easier and perhaps wider access to this literature has its own intrinsic merit and provides the necessary justification for what follows.

The Model :- Let $\mathbb{R}$ denote the set of real numbers and for any non-empty subset S of $\mathbb{R}$ let P(S) denote the set of all non-empty subsets of S. A correspondence on a non-empty subset S of $\mathbb{R}$ is a function $F : S \to P(S)$. In particular, if F is single-valued $\forall\, x \in S$, then F reduces to a function.

Let $\mathbb{N}$ denote the set of natural numbers. A correspondence F on a non-empty subset S of $\mathbb{R}$ is said to be closed if [whenever $<x_n / n \in \mathbb{N}>$ is a sequence in S with $\lim_{n\to\infty} x_n = x \in S$ and $<y_n / n \in \mathbb{N}>$ is a sequence with : (a) $y_n \in F(x_n)\ \forall\ n \in \mathbb{N}$; (b) $\lim_{n\to\infty} y_n = y \in S$], then $y \in F(x)$.

Let S be a non-empty convex subset of $\mathbb{R}$. A correspondence F on S is said to be convex valued if $\forall\ x \in S$: F(x) is a convex subset of $\mathbb{R}$. Let a and b be real numbers with a<b. Then the closed interval [a,b] is defined to be the set $\{x \in \mathbb{R} : a \leq x \leq b\}$.
The following result is a trivial consequence of the intermediate value theorem for continuous functions.

<u>Theorem</u> 1 :- Let $f : [a,b] \to [a,b]$ be a continuous function. Then there exists $x \in [a,b]$ such that $f(x) = x$.

3. <u>The One dimensional Kakutani's Fixed Point Theorem</u> :-The following consequence of Theorem 1 is referred to as the one-dimensional Kakutani's Fixed Point Theorem.

<u>Theorem</u> 2 :- Let F be a closed and convex valued correspondence on [a,b]. Then there exists $x \in [a,b]$ such that $x \in F(x)$.

<u>Proof</u> :- Suppose a = 0 and b = 1. Let m be any natural number. Let x(m,o) = 0, and having defined x(m,i) for $i \in \{0,1,\ldots,2^m-1\}$ define, $x(m,i+1) = x(m,i) + 1/2^m$. Let $T(m) = \{x(m,i) / i \in \{0,1,\ldots 2^m-1, 2^m\}\}$.
Define $f^{(1)} : [0,1] \to [0,1]$ as follows : choose, $f^{(1)}(x(1,i))$ in F(x(1,i)) for $i \in \{0,1,2\}$ and for $y \in [x(1,i), x(1,i+1)]$, let $f^{(1)}(y) = tf^{(1)}(x(1,i))+(1-t)\ f^{(1)}(x(1,i+1))$ if $y = tx(1,i)+(1-t)x(1, i+1)$ with $t \in [0,1]$.Having defined, $f^{(1)},\ldots,f^{(m-1)}$ define $f^{(m)} : [0,1] \to [0,1]$ as follows :

$f^{(m)}(y) = f^{(m-1)}(y)\ \forall\ y \in T(m-1)$;
$f^{(m)}(y) \in F(y)$ if $y \in T(m)\backslash T(m-1)$; and for $y \in [x(m,i), x(m, i+1)]$, let $f^{(m)}(y) = tf^{(m)}(x(m,i))+(1-t)f^{(m)}(x(m,i+1))$ if $y = tx(m,i) + (1-t)x(m,i+1)$ with $t \in [0,1]$.

Since each $f^{(m)} : [0,1] \to [0,1]$ is continuous, there exists $y(m) \in [0,1]$ such that $f^{(m)}(y(m)) = y(m)$. Clearly for each m, there exists x(m,i(m)) with $i(m) \in \{0,1,\ldots,2^m-1\}$ and $t_m \in [0,1]$ such that:

(i) $y(m) = t_m\, x(m,i(m)) + (1-t_m)[x(m,i(m))+ 1/2^m]$
$\quad = x(m, i(m)) + (1-t_m)[1/2^m]$
(ii) $y(m) = t_m f^{(m)}(x(m, i(m))) + (1-t_m)\ f^{(m)}(x(m,i(m)) + 1/2^m)$.

Since $\langle x(m,i(m)) / m \in \mathbf{N} \rangle$ is a sequence in [0,1] it has a convergent subsequence, which without loss of generality can be assumed to be the original sequence. Let $\lim_{m\to\infty} x(m,i(m)) = y \in [0,1]$. Since $\lim_{m\to\infty} 1/2^m = 0$, we also have, $\lim_{m\to\infty} [x(m, i(m)) + 1/2m] = \lim_{m\to\infty} y(m) = y$, recalling that $\langle 1\text{-}t_m/m \in \mathbf{N} \rangle$ is a bounded sequence. Since $\langle f^{(m)}(x(m,i(m))) / m \in \mathbf{N} \rangle$ and $\langle f^{(m)}(x(m,i(m)) + i/2^m) / m \in \mathbf{N} \rangle$ are sequences in [0,1] they both admit convergent subsequences which without loss of generality can be considered to be the original sequence (i.e. first extract a convergent subsequence from the first sequence and then extract a convergent subsequence from the corresponding subsequence of the second sequence and finally revert to the notation of the original sequence). Thus let

$\lim_{m\to\infty} f^{(m)}(x(m,i(m)) = y^1$ and

$\lim_{m\to\infty} f^{(m)}(x(m,i(m)) + 1/2^m) = y^2$.

Since F is a closed correspondence, $y^1, y^2 \in F(y)$. Further $\langle t_m/m \in \mathbf{N} \rangle$ being a sequence in [0,1] also admits a convergent subsequence which without loss of generality can be considered to be the original sequence. Let $\lim_{m\to\infty} t_m = t \in [0,1]$. Thus,

$y = \lim_{m\to\infty} y(m)$

$= \lim_{m\to\infty} t_m f^{(m)}(x(m,i(m))) + \lim_{m\to\infty} (1\text{-}t_m) f^{(m)}(x(m,i(m)) + 1/2^m)$

$= ty^1 + (1\text{-}t) y^2 \in F(y)$ since $y^1, y^2 \in F(y)$ and F is convex valued.

Now let F be a closed and convex valued correspondence on [a,b]. Define the closed and convex valued correspondence G on [0,1] as follows:

$G(x) = \{(y\text{-}a)/(b\text{-}a) / y \in F(a+x(b\text{-}a))\}$ whenever $x \in [0,1]$

By the previous argument, there exists

$x^* \in [0,1]$ such that $x^* \in G(x^*)$.

Hence there exists $y^* \in F(a + x^*(b\text{-}a))$ such that $(y^*\text{-}a)/(b\text{-}a) = x^*$

$\therefore y^* = a + x^*(b\text{-}a)$.

Thus $y^* \in F(y^*)$.

Q.E.D.

4. <u>Application</u> :- Consider an economy with n agents,where n is a positive integer, greater than or equal to two.Let 'i' denote a generic agent.Let there be two infinitely divisible goods which can be consumed in non-negative amounts only.Let the two goods be X and Y. Let $w_X(i)$ and $w_Y(i)$ be strictly positive real

numbers denoting the initial endowment of each of the two goods with agent 'i'.The preferences of agent 'i' are summarized by a utility function $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ (:where $\mathbb{R}^2_+$ denotes the non-negative orthant of two-dimensional Euclidean space).The following assumptions pertain to $u^i : \mathbb{R}^2_+ \to \mathbb{R}$, whenever $i \in N \equiv \{1,...n\}$:
(i) u^i is continuous;
(ii) u^i is weakly increasing,i.e.$a>b\geq o, c>d\geq o$ implies $u^i(a,c)>u^i(b,d)$;
(iii) u^i is quasi-concave.
A consequence of (i) and (ii) is that $a\geq b\geq o, c\geq d\geq o$ implies $u^i(a,c) \geq u^i(b,d)$.

Let the fixed price of good Y in terms of good X be equal to one.This does not lead to any loss of generality,since if the price of good Y was any other positive number all that this would entail is multiplying the units of measurement of Y by a positive scalar.Good X is treated as the numeraire good in our model. Let, $w_X = \Sigma_{i\in N} w_X(i)$ and $w_Y = \Sigma_{i\in N} w_Y(i)$ denote the aggregate quantities of the two goods in the economy.
Suppose that,with considerations of equity in mind, a social planner decides that no individual should be allowed to consume more of good X,than what would be his entitlement of good X under equal division;i.e. each individual can consume at most (w_X/n) units of good X.
Let $L_Y \geq 0$ be a constraint imposed on the individual consumption of good Y.Agent 'i' is supposed to choose his consumption bundle by solving the following optimization problem:
$u^i(x(i),y(i)) \to \max$,
s.t. $x(i)+ y(i) \leq w_X(i) + w_Y(i)$,
$0 \leq x(i) \leq (w_X/n)$, $0\leq y(i) \leq L_Y$.
Since we are maximizing a continuous function on a closed and bounded set, the above set has atleast one solution.Let $S^i(L_Y)$ denote the set of solutions for the above maximization problem.Since the utility function for each agent is quasi concave, $S^i(L_Y)$ is a convex set.Further,$\{(x(i),y(i), L_Y)/ L_Y \geq 0, (x(i),y(i)) \in S^i(L_Y)\}$ is a closed subset of $\mathbb{R}^3_+$, because the utility functions of the agents have been assumed to be continuous.

An allocation is a list $<(x(i),y(i))/i\in N>$ such that:(i) $(x(i),y(i)) \in \mathbb{R}^2_+$ for $i\in N$;(ii) $\Sigma_{i\in N}(x(i),y(i)) \leq (w_X ,w_Y)$;(iii) $\Sigma_{i\in N} y(i) = w_Y$.

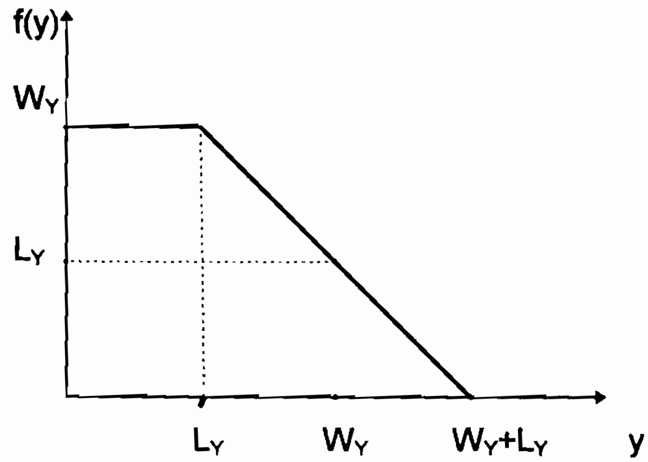
A constrained equilibrium with rationing is an ordered pair $(<(x(i),y(i))/i\in N>, L_Y)$ such that:
(i) $<(x(i),y(i))/i\in N>$ is an allocation and $L_Y \geq 0$;(ii) $\forall i\in N$:$(x(i),y(i)) \in S^i(L_Y)$;(iii) $x(i)< (w_X/n)$ implies $x(i)+ y(i) = w_X(i) + w_Y(i)$.

The following theorem is what we establish by using Theorem 2:

Theorem 3 :- Under the above assumptions,there exists a constrained equilibrium with rationing.

Proof :- Define S: $\mathbb{R}_+ \rightarrow P(\mathbb{R}^2_+)$ as follows: $\forall\ L_Y \geq 0$:$S(L_Y)= \Sigma_{i\in N}S^i\ (L_Y)$. Let S_Y: $\mathbb{R}_+ \rightarrow P(\mathbb{R}_+)$ be defined thus: $\forall\ L_Y \geq 0$: $S_Y(L_Y)$={y/there exists x such that $(x,y) \in S(L_Y)$}.Finally let F:[0, w_Y] $\rightarrow$ P([0, w_Y]) be defined as follows: $\forall\ L_Y \in [0, w_Y]$:$F(L_Y)$={max{min{ w_Y, L_Y -(y- w_Y)},0}/y$\in S_Y(L_Y)$}.Standard methods prove that F is a closed correspondence.

Given $L_Y \in [0, w_Y]$,let f: $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by f(y)= {max{min{ w_Y, L_Y -(y- w_Y)},0} whenever y$\in \mathbb{R}_+$.The following diagram makes the fact that F is convex valued self explanatory:



Thus by theorem 2, there exists $L_Y \in [0, w_Y]$ such that $L_Y \in F(L_Y)$.

Case 1:- $0 < L_Y < w_Y$.

Let y$\in S_Y(L_Y)$: L_Y = max{min{ w_Y, L_Y -(y- w_Y)},0}.

Thus, 0< max{min{ w_Y, L_Y -(y- w_Y)},0} < w_Y.

Thus, $L_Y = L_Y -(y- w_Y)$.

Thus, $y = w_Y$.

Case 2:- L_Y =0.

Let y$\in S_Y(L_Y)$: L_Y = max{min{ w_Y, L_Y -(y- w_Y)},0}.

Thus, $0 \geq$ min{ w_Y, L_Y -(y- w_Y)}.Since w_Y is poistive we get

$w_Y > 0 \geq\ -(y- w_Y)$.

But, L_Y =0 implies y = 0.

Thus, $w_Y > 0 \geq\ w_Y$ which is not possible.

Thus Case 2 is not possible.

Case 3:- $L_Y = w_Y$.

Let y$\in S_Y(L_Y)$: L_Y = max{min{ w_Y, L_Y -(y- w_Y)},0}.

Thus, $w_Y -(y- w_Y) \geq w_Y >0$.

Thus, $w_Y \geq y$.

Thus, there exists $L_Y \in (\ 0,\ w_Y\ ]$ and $y \in S_Y(L_Y)$ such that $w_Y \geq y$. Further, $w_Y > y$ implies $L_Y = w_Y$.

For $i \in N$ let $(x(i),y(i)) \in S^i(L_Y)$ so that $y = \Sigma_{i \in N} y(i)$.
Thus $x(i)+ y(i) \leq w_X(i) + w_Y(i)$ whenever $i \in N$.
Thus $x + y \leq w_X + w_Y$, where $x = \Sigma_{i \in N} x(i)$.
Since $x(i) \leq (w_X/n)$ whenever $i \in N$, clearly $x \leq w_X$.
If, $x + y = w_X + w_Y$ then $x = w_X$ and $y = w_Y$. Further, $x(i) = (w_X/n)$ whenever $i \in N$.
Thus, $(<(x(i),y(i))/i \in N>, L_Y)$ is a constrained equilibrium with rationing.
Hence suppose, $x + y < w_X + w_Y$. Suppose $y < w_Y$.
Let, $A = \{i \in N/\ x(i)+ y(i) < w_X(i) + w_Y(i)\}$. Clearly A is non-empty. Let $a(i)= [w_X(i) + w_Y(i)] - [x(i)+ y(i)]$. Let, $y^o(i) = y(i)$ if $i \in N \setminus A$ and $y^o(i) = y(i)+a(i)$ if $i \in A$. By weak monotonicity $\forall i \in N: (x(i),y^o(i)) \in S^i(L_Y)$. Since $<(x(i),y^o(i))/i \in N>$ is an allocation and $L_Y \geq 0$, we can without loss of generality assume $y = w_Y$. If $x = w_X$, then $(<(x(i),y(i))/i \in N>, L_Y)$ is a constrained equilibrium with rationing. Thus suppose, $x < w_X$. Let, $B = \{i \in N/\ x(i) < (w_X/n)\}$. For $i \in N$, let (i) $x^o(i)= \min \{(w_X/n), x(i)+a(i)\}$ if $i \in A \cap B$; (ii) $x^o(i) = x(i)$, otherwise. By weak monotonicity $\forall i \in N: (x^o(i),y(i)) \in S^i(L_Y)$. Since, (i) $<(x^o(i),y(i))/i \in N>$ is an allocation and $L_Y \geq 0$; (ii) $x^o(i) < (w_X/n)$ implies $x^o(i)+ y(i) = w_X(i) + w_Y(i)$, it follows that $(<(x^o(i),y(i))/i \in N>, L_Y)$ is a constrained equilibrium with rationing.

Q.E.D.

Suppose in addition to the assumptions invoked in Theorem 3,we assume thefollowing: $\forall\ i \in N: (w_X(i), w_Y(i))= (w_X/n, w_Y/n)$. Then, there exists a constrained equilibrium with rationing $(<(x(i),y(i))/i \in N>, L_Y)$ such that, $\forall\ i \in N: x(i)= w_X/n$, $y(i)= w_Y/n$ and $L_Y == w_Y/n$.

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