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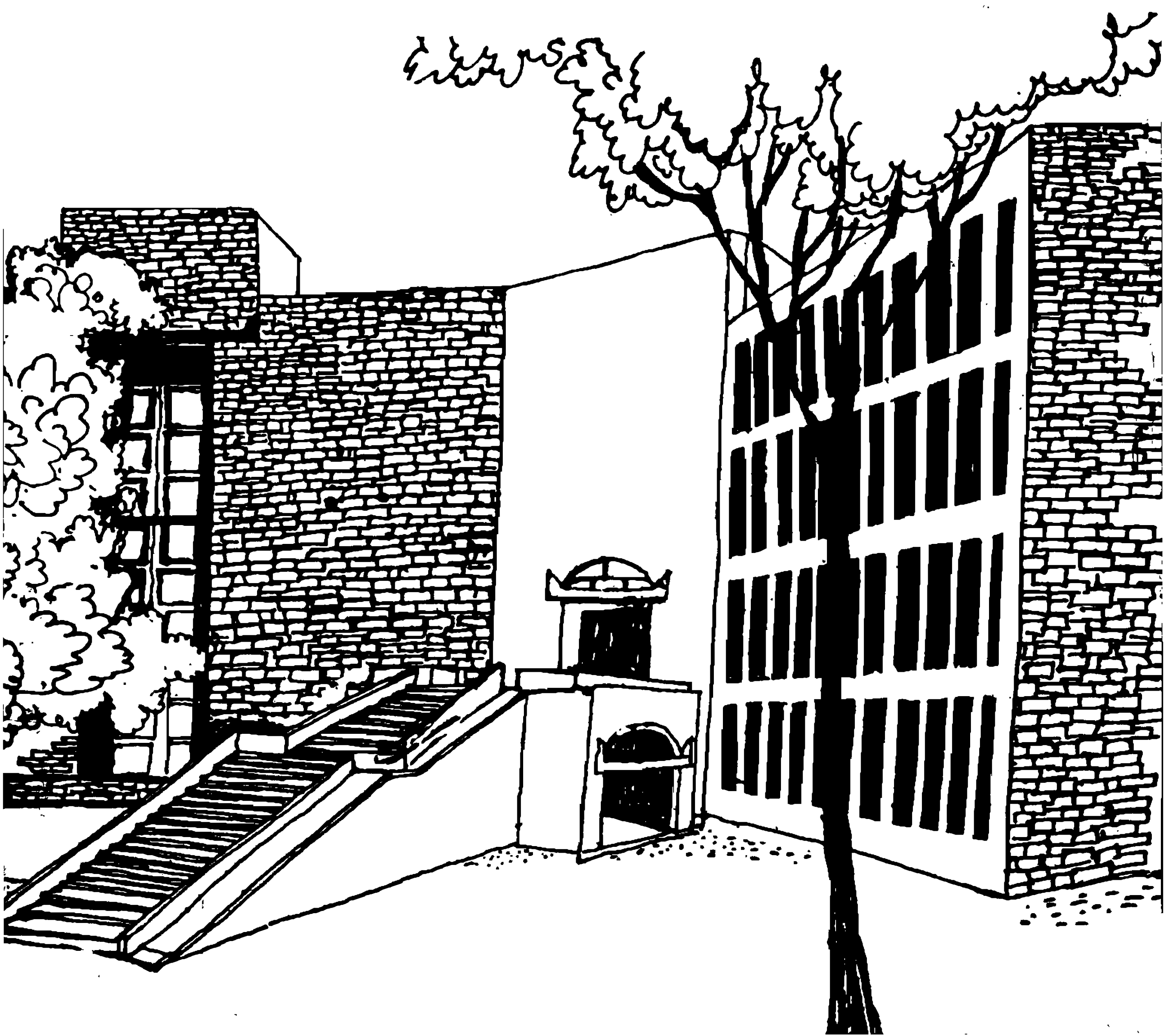

A CONSEQUENCE OF CHERNOFF AND OUTCASTING
AND SOLUTIONS FOR ABSTRACT GAMES

By

Somdeb Lahiri

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Abstract of :

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The purpose of this paper is to prove by induction the theorem (in Aizerman and Malishevski [1981]) that a choice funtion which satisfies Chernoff's axiom and Outcasting can always by expressed as the union of the solution sets of a finite number of maximization problems. In this paper we also show that the Slater solution for abstract games (see Slater [1961]) satisfies the Chernoff,Outcasting and Expansion axioms.On the other hand the solution due to Copeland [1951] , which has subsequently been axiomatically characterized by Henriet [1985],does not satisfy any of these three properties.

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Introduction

The purpose of this paper is to prove by induction the theorem (in Aizerman and Malishevski [1981]) that a choice funtion which satisfies Chernoff's axiom and Outcasting can always by expressed as the union of the solution sets of a finite number of maximization problems. The proof we offer is considerably simpler than the one in Aizerman and Malishevski [1981]. In Moulin [1985], a discussion of a similar result is available. Our framework closely resembles the one of choice theory as enunciated in Moulin [1985]. It is well known that a combination of Chernoff's axiom and Outcasting is equivalent to a property called Path Independence (See Moulin [1985]).

The idea of a function which associates with each set and a binary relation a non-empty subset of the given set has a long history whose exact origin is very difficult to specify and in any case is unknown to the author.In Laslier [1997] can be found a very exhaustive survey of the related theory when binary relations are reflexive,complete and anti-symmetric.

In a related paper (Lahiri [2000a])we extend the above set of binary relations to include those which are not necessarily anti-symmetric.Such binary relations which are reflexive and complete are referred to in the literature as abstract games. An ordered pair comprising a non-empty subset of the universal set and an abstract game is referred to as a subgame.A (game)solution is a function which associates to all subgames of a given (nonempty) set of games,a nonempty subset of the set in the subgame.Lucas [1992] has a discussion of abstract games and related solution concepts, particularly in the context of cooperative games.Moulin [1986],is really the rigorous starting point of the axiomatic analysis of game solutions defined on tournaments,i.e.anti-symmetric abstract games.Much of what is discussed in Laslier [1997] and references therein carry through into this framework.In Lahiri [2000 b],we obtaian necessary and sufficient conditions that an abstract game needs to satisfy so that every subgame has atleast one von Neumann-Morgenstern stable set.

In the final section of this paper we show that the Slater solution for abstract games (see Slater [1961]) satisfies the Chernoff,Outcasting and Expansion axioms.On the other hand the solution due to Copeland [1951] ,which has subsequently been axiomatically characterized by Henriet [1985],does not satisfy any of these three properties.

The Framework

Let X be a finite, non empty universal set. If A is any non-empty subset of X, let [A] denote the set of all non-empty subsets of A. A choice function on X is a function $C:[X] \to [X]$ such that $C(A) \subset A \; \forall A \in [X]$.

Given $A \in [X]$, let $|A|$ denote the cardinality of A. C is said to satisfy:

a) Chernoff Axiom (CA), if $\forall$ A, B $\in$ [X], A $\subset$ B implies C (B) $\cap$ A $\subset$ C (A) ;

b) Outcasting (O) , if $\forall$ A, B $\in$ [X], C(B) $\subset$ A $\subset$ B implies C(B) = C(A).

c) Aizerman (A), if $\forall$ A, B $\in$ [X], C(B) $\subset$ A $\subset$ B implies C(A) $\subset$ C(B).

Chernoff Axiom was originally proposed in Chernoff [1954]. Outcasting, which occurs under its present nomenclature in Aizerman and Aleskerov [1995], has been attributed to Nash [1950],by Suzumura [1983]. Aizerman has been in the literature for a while (for example,see Fishburn [1975]). However, its prominent role was recognized only recently (Aizerman and Malishevsky [1981]).

Clearly, Outcasting implies Aizerman. It is also quite easy to see that Aizerman and Chernoff together imply Outcasting. Hence, a choice function satisfies Aizerman and Chernoff if and only if it satisfies Outcasting and Chernoff.

The issue here is the following theorem in Aizerman and Malishevski [1981] :

Theorem 1: Let C be a choice function on X which satisfies CA and O. Then there exists n $\in \mathbf{N}$ and functions $f_i : X \to \mathbf{N}$, i $\in$ {1, ..., n} such that $\forall$ A $\in$ [X],

$$C(A) = \bigcup_{i=1}^{n} \{x \in A \ / \ f_i(x) \geq f_i(y) \ \forall \ y \in A\}$$

Before we provide a new proof of this theorem, let us provide two examples to show that neither CA nor O is alone sufficient for the above theorem.

Example 1 : Let X = { x,y,z}, C(X) = {x}, and C(A) = A $\forall$ A $\in$ [X], A $\subset\subset$ X. Clearly C satisfies CA but not O Towards a contradiction suppose there exists n $\in \mathbf{N}$ and functions $f_i : X \to \mathbf{N}$, i = 1, ..., n such that

$$C(A) = \bigcup_{i=1}^{n} \{a \in A \ / \ f_i(a) \geq f_i(b) \ \forall \ b \in A\} \ \forall \ A \in [X].$$

Then $C(X) = \{x\}$ implies $f_i(x) > \max \{f_i(y), f_i(z)\} \ \forall \ i$.

However, $C(\{x,y\}) = \{x,y\}$ implies $f_i(y) \geq f_i(x)$ for some i, which condtradicts what we obtained before.

<u>Example 2</u> : Let $X = \{x,y,z\}$, $C(X) = X$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\} \ \forall \ a \in X$. Clearly C satisfies O but not CA. Towards a contradiction suppose there exist $n \in \mathbb{N}$ and functions $f_i : X \to \mathbb{N}$, $i = 1, \ldots, n$ such that

$$C(A) = \bigcup_{i=1}^{n} \{a \in A \ / \ f_i(a) \geq f_i(b) \ \forall \ b \in A\} \ \forall \ A \in [X].$$

Then $C(X) = X$ implies there exists $i \in \{1, \ldots, n\}$ such that $f_i(y) \geq f_i(x)$. However, then $y \in C(\{x,y\})$, contrary to our definition of C.

<u>Proof of Theorem 1</u> :

We will prove this theorem by induction on the Cardinality of X.

If $|X| = 2$, then there are two possibilities :

a) $C(X) = X$: then define $f : X \to \mathbb{N}$ as follows :

$f(a) = 1 \forall \ a \in X$.

b) $C(X) \neq X$: then define $f : X \to \mathbb{N}$ as follows :

$f(a) = 2$ if $a \in C(X)$

$= 1$ if $a \in X \setminus C(X)$.

Clearly $C(A) = \{a \in A \mid f(a) \geq f(b) \ \forall \ b \in A\}$.

Hence suppose the theorem is true for $|X| \in \{1,\ldots,m-1\}$ and suppose $|X| = m \in \mathbb{N}$. Let $C(X) = \{x_1, \ldots, x_p\}$, for some $p \in \mathbb{N}$. For each $x_i \in C(X)$, let $Y_i = X \setminus \{x_i\}$. Then

$\forall (\emptyset \neq) A \subset B \subset Y_i$, $C(B) \cap A \subset C(A)$

$\forall (\emptyset \neq) A \subset B \subset Y_i$, if $C(B) \subset A$ then $C(A) = C(B)$.

Let $C_i : [Y_i] \to [Y_i]$ be defined as follows :

$C_i(A) = C(A)\ \forall A \in [Y_i]$, $i \in \{1, \ldots, p\}$.

By the induction hypothesis $\forall i \in \{i, \ldots, p\}$, there exists $m_i \in \mathbb{N}$ and $g_i^j : Y_i \to \mathbb{N}$, $j = 1,\ldots, m_i$ such that

$$C_i(A) = \bigcup_{j=1}^{m_i} \{a \in A / g_i^j(a) \geq g_i^j(b) \forall b \in A\}, \forall A \in [Y_i].$$

Let $g_i^j(x_i) = [\max\{g_i^j(a) / a \in Y_i\}] + 1$,

$\forall j \in \{1,\ldots,m_i\}$, $i \in \{1, \ldots, p\}$.

Now suppose $A \in [X]$.

Suppose $A \subset Y_i\ \forall i \in \{1,\ldots,p\}$.

Then $C(A) = C_i(A) = \bigcup_{j=1}^{m_i} \{a \in A / g_i^j(a) \geq g_i^j(b) \forall b \in A\}\ \forall i \in \{1,\ldots,p\}$.

$$\therefore C(A) = \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in A / g_i^j(a) \geq g_i^j(b) \forall b \in A\}.$$

Hence suppose $A \not\subset Y_i$ for some $i \in \{i,\ldots,p\}$.

<u>Case 1</u> : $C(X) \subset A$

Then, by (O), $C(A) = C(X)$.

$$\therefore C(A) = \{x_1,\ldots,x_p\} = \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in A \mid g_i^j(a) \geq g_i^j(b) \; \forall\, b \in A\}.$$

<u>Case 2</u> : $C(X) \not\subset A$.

Let $A = \{i \mid x_i \notin A\} \neq \emptyset$

Thus $A \subset Y_i \; \forall\, i \in A$.

By the induction hypothesis,

$$C(A) = C_i(A) = \bigcup_{j=1}^{m_i} \{a \in A \mid g_i^j(a) \geq g_i^j(b) \; \forall\, b \in A\}, \forall\, i \in A.$$

Hence,

$$C(A) \subset \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in A \mid g_i^j(a) \geq g_i^j(b) \; \forall\, b \in A\}.$$

Now suppose $i \notin A$. Thus $x_i \in C(X) \cap A$. By CA, $x_i \in C(A)$

$$\therefore \bigcup_{i \notin A} \bigcup_{j=1}^{m_i} \{a \in A \mid g_i^j(a) \geq g_i^j(b) \; \forall\, b \in A\} \subset C(A).$$

$$\text{But, } C(A) = C_i(A) = \bigcup_{j=1}^{m_i} \{a \in A \mid g_i^j(a) \geq g_i^j(b) \; \forall\, b \in A\}, \forall\, i \in A.$$

$$\therefore \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in A \mid g_i^j(a) \geq g_i^j(b) \; \forall\, b \in A\} \subset C(A).$$

$$\text{Hence } C(A) = \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in A \mid g_i^j(a) \geq g_i^j(b) \; \forall\, b \in A\}, \forall\, A \in [X].$$

The theorem was shown to hold for $|X| = 2$ and has now been shown to hold for $|X| = m$ if it holds for $|X| = m-1$. Hence it is true for all finite non-empty X.

Q.E.D.

<u>Remark</u>: In Moulin [1985],there is a property called Expansion. C is said to satisfy Expansion (E) , if $\forall\, A, B \in [X]$, $C(B) \cap C(A) \subset C(A \cup B)$.

The result due to Schwarz [1976],which we refer to in the introduction as the one available in Moulin [1985] implies the following:

Let C be a choice function on X which satisfies CA ,E and O. Then there exists n $\in \mathbb{N}$ and functions $f_i : X \to \mathbb{N}$, $i \in \{1, ..., n\}$ such that $\forall A \in [X]$,

(1) $C(A) = \bigcup_{i=1}^{n} \{x \in A \,/\, f_i(x) \geq f_i(y) \ \forall \ y \in A\}$ and (2) $C(A) = \{x \in A / x \in C(\{x,y\}) \forall y \in A\}$.Conversely (1) and (2) imply C satisfies CA,E and O.

The following example shows that (1) above may be satisfied even if C does not satisfy E.

<u>Example 3</u> : Let X = {x,y,z}, C(X) = {y,z}, C({x,y}) = {x,y}, C({y,z}) = {y,z}, C ({x,z}) = {x,z}, C({a}) = {a} $\forall$ a $\in$ X. Clearly C satisfies CA and O but not E, since $x \in C(\{x,a\}) \forall a \in X$ and yet $x \notin C(X)$. Let $f_i : X \to \mathbb{N}$, i = 1, 2 be such that $f_1(y)=3> f_1(x)=2> f_1(z)=1$ and $f_2(z)=3> f_2(x)=2> f_2(y)=1$. However,

$$C(A) = \bigcup_{i=1}^{2} \{a \in A \,/\, f_i(a) \geq f_i(b) \ \forall \ b \in A\} \ \forall \ A \in [X].$$

<u>Quasi-Transitive Binary Relations</u>

A binary relation Q on X is any non-empty subset of X x X. Given a binary relation Q on X its <u>asymmetric</u> part denoted P(Q) = {(x, y) $\in$ Q/ (y, x) $\notin$ Q } and the <u>symmetric part</u> of Q denoted I(Q)= {(x, y) $\in$ Q/ (y, x) $\in$ Q }.A binary relation Q on X is said to be

(i) reflexive if (x, x) $\in$ Q $\forall$ x $\in$ X;

(ii) complete if x, y $\in$ X, x $\neq$ y implies (x, y) $\in$ Q or (y, x) $\in$ Q;

(iii) quasi-transitive if $\forall$ x, y, z $\in$ X, (x, y) $\in$ P (Q) and (y, z) $\in$ P(Q) implies (x,z) $\in$ P(Q);

(iv) a quasi order if it is reflexive,complete and quasi-transitive.

We are concerned here with the following theorem, which may be found in Roberts [1979], Aizerman and Malishevsky [1981],Moulin [1985] (and which has been generalized in Lahiri [1999] to the case where the universal set X is possibly infinite) and which now follows as an easy corollary of our Theorem 1:

Theorem 2: Q is a quasi order on X if and only if there exists a positive integer n and functions $f_i:X \to \mathbb{N}$, $i \in \{1, ..., n\}$ such that $Q = \{ (x, y) \in X \times X / f_i(x) \geq f_i(y)$ for some $i \in \{1, ..., n\}\}$.

Proof:- It is easy to see that if there exists a positive integer n and functions $f_i:X \to \mathbb{N}$, $i \in \{1, ..., n\}$ such that $Q = \{ (x, y) \in X \times X / f_i(x) \geq f_i(y)$ for some $i \in \{1, ..., n\}\}$ then Q is a quasi order.To prove the converse assume that Q is a quasi order. For $A \in [X]$, let $C(A) =\{x \in A / (x,y) \in Q\ \forall\ y \in X\}$.Clearly $C(A) \neq \phi$ whenever $A \in [X]$, since Q is a quasi order. Hence C as defined above is a choice function.Further it is easy to verify that C satisfies CA and O. Hence,by Theorem 1, there exists a positive integer n and functions $f_i : X \to \mathbb{N}$ for $i \in \{1, .., n\}$, such that $C(A) = \bigcup_{i=1}^{n} \{x \in A \ / \ f_i(x) \geq f_i(y)\ \forall\ y \in A\}\ \ \forall A \in [X]$. Since $(x, y) \in Q$ if and only if $x \in C(\{x,y\})$, and since $x \in C(\{x,y\})$ if and only if $f_i(x) \geq f_i(y)$ for some $i \in \{1, ..., n\}$,the proof of the theorem is thereby complete.

Q.E.D.

Stronger Consequences

The following lemma permits to strengthen the two theorems obtained above:

Lemma 1 : Let $f:X \to \Re$ (:the set of real numbers) be given. Then, there exists a positive integer n and one to one functions $f_i:X \to \mathbb{N}$, $i \in \{1, ..., n\}$ such that

$\{ (x, y) \in X \times X / f(x) \geq f(y)\} = \{ (x, y) \in X \times X / f_i(x) \geq f_i(y)$ for some $i \in \{1, ..., n\}\}$.

Proof :- Let $\{f(x)/ x \in X\} = \{s_1, ..., s_q\}$ where q is a positive integer and $s_j < s_{j+1}$ $\forall j \in \{1, ..., q-1\}$.Let $n_j = |\{x \in X/ f(x)= s_j\}|$ and let $n= (n_1)! \times ... \times (n_q)!$

Let $g:X \to \mathbf{N}$ be defined as follows:

$g(x) = n_1$, if $f(x) = s_1$

$g(x) = n_1 + ... + n_j$,if $f(x) = s_j$

Clearly, $\forall x,y \in X$: [$f(x) \geq f(y)$ if and only if $g(x) \geq g(y)$].

A function $\pi : \{1,..., n_1 + ... + n_q\} \to X$ is called a restricted permutation if $\forall$ k $\in \{1,..., n_1 + ... + n_q\}$: (1) [$\pi(k) \in \{x \in X/ f(x)= s_1\}$ if and only ($1 \leq k \leq n_1$)] & (2) [$\pi(k) \in \{x \in X/ f(x)= s_i\}$ if and only ($n_{i-1} \leq k \leq n_i$ and $1 < i \leq q$)].Let Π denote the set of all restricted permutations.Since X is finite so is Π. For $\pi \in \Pi$, define $f_\pi : X \to \{1,..., n_1 + ... + n_q\}$ as follows: $\forall x \in X$, $f_\pi(x) = k$ if and only if $\pi(k) = x$. It is now easy to verify that, $\{ (x, y) \in X \times X/f(x) \geq f(y)\} = \{ (x, y) \in X \times X/g(x) \geq g(y)\} = \{ (x, y) \in X \times X/f_\pi(x) \geq f_\pi(y)$ for some $\pi \in \Pi\}$. This proves the lemma.

Q.E.D.

In view of Lemma 1 and Theorems 1 and 2 we have the following:

Theorem 3: Let C be a choice function on X which satisfies CA and O. Then there exists $n \in \mathbf{N}$ and one to one functions $f_i : X \to \mathbf{N}$, $i \in \{1, ..., n\}$ such that

$$\forall A \in [X],\ C(A) = \bigcup_{i=1}^{n} \{x \in A \ / \ f_i(x) \geq f_i(y) \ \forall \ y \in A\}.$$

Theorem 4: Q is a quasi order on X if and only if there exists a positive integer n and one to one functions $f_i:X \to \mathbf{N}$, $i \in \{1, ..., n\}$ such that $Q = \{ (x, y) \in X \times X/f_i(x) \geq f_i(y)$ for some $i \in \{1, ..., n\}\}$.

Game Solutions

A binary relation R on X is said to be transitive if $\forall$ x, y, z $\in$ X, [(x, y) $\in$ R & (y, z) $\in$ R implies (x, z) $\in$ R] and it is said to be anti-symmetric if [$\forall$ x, y $\in$ X, (x, y)

$\in R$ & $(y, x) \in R$ implies $x = y$]. Given a binary relation R on X and $A \in [X]$, let $R|A = R \cap (A \times A)$.

Let Π denote the set of all reflexive and complete binary relations. If $R \in \Pi$, then R is called an abstract game. An ordered pair $(A,R) \in [X] \times \Pi$ is called a subgame. Given a binary relation R on X and $A \in [X]$, let $G(A,R) = \{ x \in A / \forall y \in A : (x,y) \in R\}$. Given $A \in [X]$, let $\Delta(A)$ denote the diagonal of A i.e. $\Delta(A) \equiv \{(x,x)/x \in X\}$.

The following example shows that given $R \in \Pi$ and $A \in [X]$, G(A,R) may be empty:

Example 4:Let $X = \{x,y,z\}$ and let $R = \Delta(X) \cup \{(x,y),(y,z),(z,x)\}$.Clearly G(X,R) is empty.

Let,Λ be a non-empty subset of Π.

A (game) solution on Λ is a function $S: [X] \times \Lambda \to [X]$ such that:

(i) $\forall (A,R) \in [X] \times \Lambda : S(A,R) \subset A$;

(ii) $\forall (A,R), (A,Q) \in [X] \times \Lambda : R|A = Q|A$ implies $S(A,R) = S(A,Q)$.

Let S be a solution on Λ and let $R \in \Lambda$.Let $S(R):[X] \to [X]$ be defined thus:$\forall A \in [X] : S(R)(A) = S(A,R)$.Clearly S(R) is a choice function.

If $\forall (A,R) \in [X] \times \Lambda$, G(A,R) is non-empty valued then the associated solution is called the best solution on (X, Λ).

Given an abstract game R,it is said to be a transitive abstract game, if R is a transitive binary relation.Let Ω be the set of transitive abstract games.It is wellknown that $R \in \Pi$ if and only if there exists a function $f: X \to \Re$ such that $\forall x,y \in X : (x,y) \in R$ if and only if $f(x) \geq f(y)$.

The Hamming distance on Π denoted $H: \Pi \times \Pi \to \Re$ (or simply H) is defined as follows: $H(R,Q) = |R \backslash Q| + |Q \backslash R|$.It is easy to see that H is a metric on Π.

Given $R \in \Pi$,let $\Omega(R) = \{Q \in \Omega / \forall Q' \in \Omega : H(R,Q) \leq H(R,Q')\}$.

Example 5 :Let $X=\{x,y,z\}$ and let $R=\Delta(X)\cup\{(x,y),(y,z),(z,x)\}$.Let $Q_1=(R\cup\{(x,z)\})\backslash\{(z,x)\}$, $Q_2=(R\cup\{(y,x)\})\backslash\{(x,y)\}$, $Q_3=(R\cup\{(z,y)\})\backslash\{(y,z)\}$.$\forall i\in\{1,2,3\}$: $H(R,Q_i)=2$.Towards a contradiction suppose that Q is a transitive game with $H(R,Q)<2$.$H(R,Q)>0$,since R itself is not transitive.Hence suppose that $H(R,Q)=1$.Thus either $Q\subset\subset R$ and $|R\backslash Q|=1$or $R\subset\subset Q$ and $|Q\backslash R|=1$.If $Q\subset\subset R$ then Q cannot be complete and thus Q is not a transitive game.Thus, $R\subset\subset Q$ and $|Q\backslash R|=1$.But then Q is not transitive.Hence $\Omega(R)=\{Q_1,Q_2,Q_3\}$.

We have thus established the following:

Proposition 1: There exists an abstract game R such that $\Omega(R)$ contains more than one element.

The Slater solution SL: $[X]\times\Pi\to[X]$ is defined as follows:$\forall(A,R)\in[X]\times\Pi$:$SL(A,R)=\cup\{G(A,Q)/Q\in\Omega(R)\}$.

A solution S on a non-empty subset Λ of Π is said to be a Slater selection if for all R in Λ there exists Q in $\Omega(R)$(:possibly depending on R) such that for all A in [X]:$S(A,R)=G(A,Q)$.A Slater selection is by its definition a very well-behaved solution.

A solution S on a non-empty subset Λ of Π is said to satisfy:

a) Chernoff Axiom (CA^*), if $\forall R\in\Lambda$: S(R) satisfies CA;

b) Outcasting (O^*) , if $\forall R\in\Lambda$: S(R) satisfies O;

c) Expansion (E^*) , if $\forall R\in\Lambda$: S(R) satisfies E.

By Theorem 1, SL satisfies both $CA^*$and O^*.It may be of some interest to find out whether SL satisfies E^*.However,we can prove the following:

Proposition 2: There exists a solution S on Π different from SL which satisfies CA^* and O^*.

Proof :-Let $X=\{x,y,z\}$ and $Q=X\times X$.For $R\in\Pi\backslash\{Q\}$,let $S(A,R)=SL(A,R)$.Let $S(X,Q)=\{x,y\}$ and let $S(A,Q)=A$ otherwise.Clearly S satisfies $CA^*$and O^* ,although $S\neq SL$.

Q.E.D.

Proposition 3 :- Let $R \in \Pi$.Then there does not exist Q,Q' in $\Omega(R)$ and x,y,z in X, such that : (a) $\{(x,y),(y,z),(x,z)\} \subset Q \subset \Delta(X) \cup \{(x,y),(y,z),(z,y),(x,z)\}$;
(b) $\{(z,y),(y,x),(z,x)\} \subset Q' \subset \Delta(X) \cup \{(z,y),(y,x),(x,y),(z,x)\}$.

Proof :- Suppose towards a contradiction that there exists Q,Q' in $\Omega(R)$ and u,v,w in X, such that :(a) $\{(u,v),(v,w),(u,w)\} \subset Q \subset \Delta(X) \cup \{(u,v),(v,w),(w,v),(u,w)\}$; (b) $\{(w,v),(v,u),(w,u)\} \subset Q' \subset \Delta(X) \cup \{(w,v),(v,u),(u,v),(w,u)\}$. Suppose without loss of generality that $X = \{x,y,z\}$.Thus, $H(Q,Q') \geq 4$.If R is a transitive abstract game then clearly the above is not possible since then $\Omega(R) = \{R\}$. Hence suppose that R is not transitive.By the triangle inequality,$H(R,Q)=H(R,Q') \geq 2$.

Case 1:$(x,y) \in P(R)$, $(y,z) \in P(R)$ and $(z,x) \in R$.In this case $\Omega(R)$ is either a singleton (i.e. if $(z,x) \in I(R)$) or as in Example 5, contrary to the above .

Case 2:$(x,y) \in P(R)$, $(y,z) \in I(R)$ and $(z,x) \in R$. In this case $\Omega(R) = \{R \setminus \{(y,z)\}\}$, which is also a singleton and the above situation cannot arise.

Case 3: $(x,y) \in I(R)$, $(y,z) \in P(R)$ and $(z,x) \in R$. In this case $\Omega(R) = \{R \setminus \{(x,y)\}\}$, which is also a singleton and the above situation cannot arise.

Case 4: $(x,y) \in I(R)$, $(y,z) \in I(R)$ and $(z,x) \in P(R)$. In this case $R \cup \{(x,z)\} \in \Omega(R)$, and $H(R, R \cup \{(x,z)\}) = 1 < 2$.Hence the above situation cannot arise.

Case 5: $(x,y) \in I(R)$, $(y,z) \in I(R)$ and $(z,x) \in P(R)$. In this case $R \cup \{(x,z)\} \in \Omega(R)$, and $H(R, R \cup \{(x,z)\}) = 1 < 2$.Hence the above situation cannot arise.

Case 6: $(x,y) \in I(R)$, $(y,z) \in I(R)$ and $(x,z) \in P(R)$. In this case $R \cup \{(z,x)\} \in \Omega(R)$, and $H(R, R \cup \{(z,x)\}) = 1 < 2$.Hence the above situation cannot arise.

This proves the proposition.

Q.E.D.

The following is obtained as an easy consequence of Proposition 2:

Theorem 5 : SL satisfies E^*.

Proof :Let $(A,R) \in [X] \times \Pi$. By proposition 3, if $|A| \leq 3$ then $G(A, \cup\{Q \in \Omega(R)\}) \subset \cup\{G(A,Q)/Q \in \Omega(R)\}$. Suppose that if $|A| \leq K$, then $G(A, \cup\{Q \in \Omega(R)\}) \subset \cup\{G(A,Q)/Q \in \Omega(R)\}$.Let $|A| = K+1$, and towards a contradiction suppose that $G(A, \cup\{Q \in \Omega(R)\}) \not\subset \cup\{G(A,Q)/Q \in \Omega(R)\}$. Thus there exists $y \in G(A, \cup\{Q \in \Omega(R)\}) \setminus$

$\cup\{G(A,Q)/Q\in\Omega(R)\}$.If $\cup\{G(A,Q)/Q\in\Omega(R)\}$ is a singleton then clearly $G(A,\cup\{Q\in\Omega(R)\}) = \cup\{G(A,Q)/Q\in\Omega(R)\}$.Hence let $x,z \in \cup\{G(A,Q)/Q\in\Omega(R)\}$ with $x \neq z$. By the induction hypothesis, $y \in (\cup\{G(A \setminus\{x\},Q)/Q\in\Omega(R)\})\cap (\cup\{G(A \setminus\{z\},Q)/Q\in\Omega(R)\})$. Let $Q_1 \in\Omega(R)$ such that $y \in G(A \setminus\{x\},Q_1)$ and let $Q_2 \in\Omega(R)$ such that $y \in G(A \setminus\{z\},Q_2)$. Clearly $(x,y) \in P(Q_1),(y,z) \in Q_1$,$(z,y) \in P(Q_2)$ and $(y,x) \in Q_2$. This contradicts the conclusion of Proposition 3 Hence, $G(A,\cup\{Q\in\Omega(R)\}) \subset \cup\{G(A,Q)/Q\in\Omega(R)\}$.By a standard induction argument $G(A,\cup\{Q\in\Omega(R)\}) \subset \cup\{G(A,Q)/Q\in\Omega(R)\}$ for all A in [X]. This in conjuction with CA^*, which SL satisfies proves the theorem.

Q.E.D.

Given $R \in \Pi$, $A \in [X]$ and $x \in X$ let $s(x,A,R)=|\{y\in A/(x,y) \in P(R)\}|-|\{y\in A/(y,x) \in P(R)\}|$.

The Copeland solution Co: $[X]\times\Pi\rightarrow[X]$ is defined as follows:

$\forall (A,R) \in[X]\times \Pi$:$Co(A,R)=\{x\in A/\forall y\in A: s(x,A,R) \geq s(y,A,R)\}$.

<u>Proposition 4</u>:(a)Co does not satisfy CA^* ;(b)Co does not satisfy O^*;(c)Co does not satisfy E^*;(d)there exists R such that G(A,R) is not a subset of Co(A,R) for some A in [X].

<u>Proof</u> :-Let $X=\{x,y,z\}$.(a) Let $R =\Delta (X)\cup\{ (x,y),(y,z),(z,x)\}$.Now,Co(X,R)=X and $y\in Co(X,R)\cap\{x,y\}$.However $y\notin Co(\{x,y\},R)$.Thus Co does not satisfy CA^*.(b) Let $R =\Delta (X)\cup\{ (x,y),(y,x),(y,z),(z,x),(x,z)\}$. $Co(X,R)=\{y\}\subset\{x,y\}$. However, $Co(\{x,y\})=\{x,y\}\neq\{y\}= Co(X,R)$.Thus Co does not satisfy O^*; (c)Let R be as in (b). Now $x \in Co(\{x,y\},R) \cap Co(\{x,z\},R)$. However, $Co(X,R)= \{y\}$.Thus Co does not satisfy E^*.(d)Let R be as in (b) and (c).$x\in G(X,R)$ but $Co(X,R)= \{y\}$.Thus G(X,R) is not a subset of Co(X,R).

Q.E.D.

Thus the Copeland solution apart from not satisfying either CA^* or O^* , fails other tests that a desirable solution may be required to satisfy.

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