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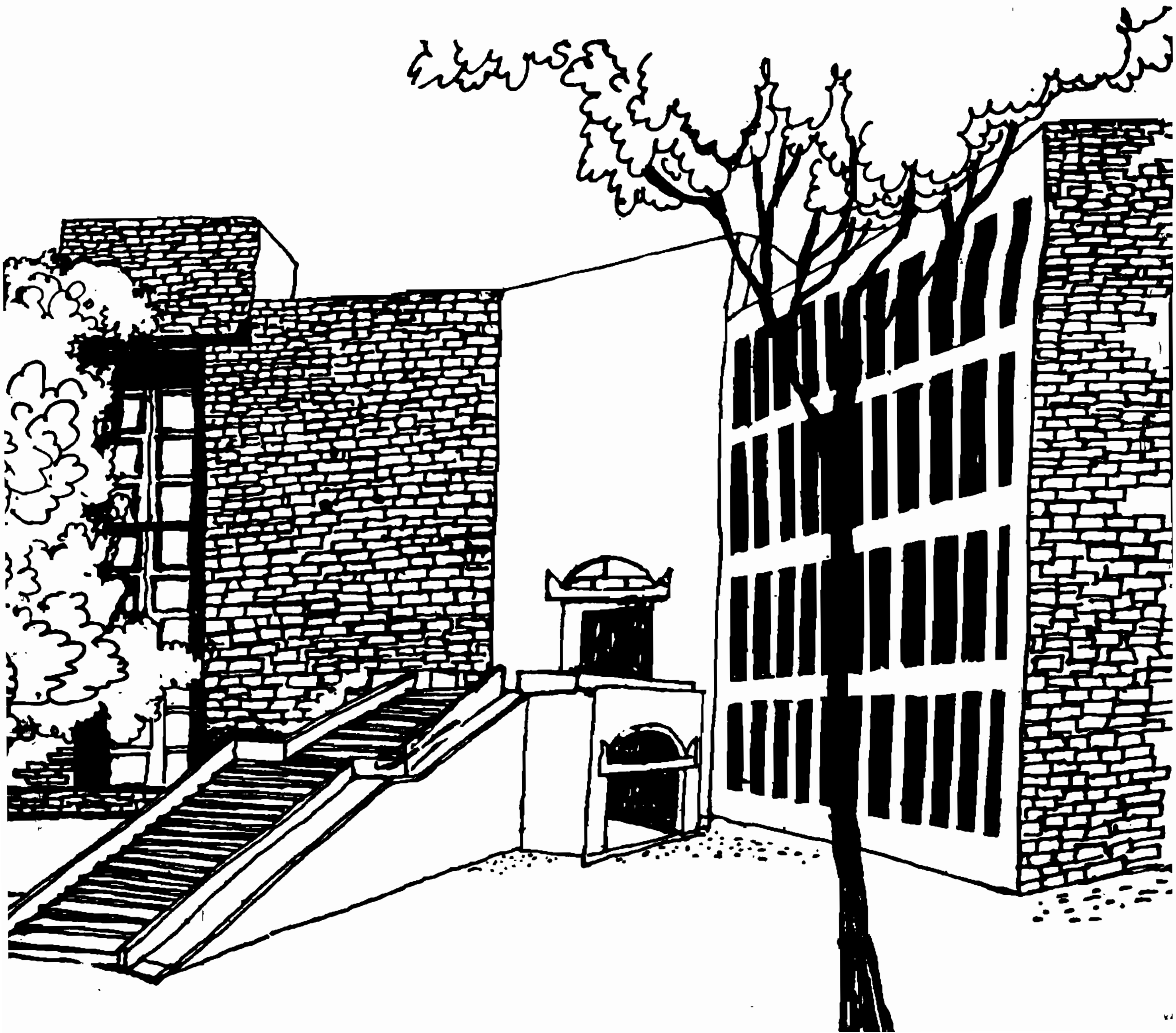

QUASITRANSITIVITY AND MONOTONIC PREFERENCE FOR FREEDOM

By

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AND
MONOTONIC PREFERENCE FOR FREEDOM

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ABSTRACT

We consider a finite universal set of alternatives and the set of all feasible sets are simply the set of all non-empty subsets of this universal set. A choice function assigns to each feasible set a non-empty subset of it.In such a framework we propose and study necessary and sufficient conditions for quasi transitive rationalizability.In a final section of this paper,we analyse necessary and sufficient conditions for quasi transitive rationalizability of choice functions generated by a monotonic preference for freedom.

1.Introduction: We consider a finite universal set of alternatives and the set of all feasible sets are simply the set of all non-empty subsets of this universal set. A choice function assigns to each feasible set a non-empty subset of it.

An interesting problem in such a context is to explore the possibility of the choice function coinciding with the best elements with respect to a binary relation. This is precisely the problem of rational choice theory. There is a large literature today on this topic.A comprehensive survey of the major results in this area (:upto the mid-eighties) is available in Moulin (1985).

In this paper, we propose a new axiom which is used to fully characterize all choice functions which are rationalized by quasi-transitive binary relations. These "almost" transitive (: but not exactly so!) binary relations, which are now quite popular in the literature (: see Yu [1985]), have the rather interesting feature of revealing intransitive indifference for single valued choice functions. This phenomena has been dealt with rather elegantly by Kim [1987]. Our purpose, is to shed new light on the problem in the absence of the single-valuedness assumption. We, propose an axiomatic characterization which is minimal. Several examples are provided, to show that the assumptions we use are logically independent.

In a final section of this paper we address the problem concerning quasi-transitive rationalizations of choice functions generated by what we refer to in this paper as "preference for freedom".The concept of "preference for freedom",can be traced back to the modest yet significant literature on "freedom of choice".In the "freedom of choice" literature, the principal problem is to define a binary relation on non-empty subsets of a given set, so as to formalize the notion of "preference for freedom" which any non-empty set of alternatives provides to a decision maker. Presumably, the idea is to use this binary relation to rank opportunity sets and arrive at decisions on the basis of such a ranking. This field has been pioneered by Pattanaik and Xu [1990], with subsequent contributions by Pattanaik and Xu [1997, 1998], Arrow [1995], Carter [1996], Puppe [1996], Sen [1990, 1991], Rosenbaum [1996],Van Hees [1998, 1999] , Van Hees and Wissenburg [1999] and Arlegi and Nieto[1999] (as also the references therein). In a related effort (Lahiri[1999]) a necessary and sufficient condition has been proposed which answers the question arising out of the converse problem: given a choice function, is there anything akin to a "preference for freedom" (: however, queer that may be !) which rationalizes the observed behaviour of a decision maker? Puppe[1996], considers a choice function which chooses only those points from a feasible set,whose unilateral deletion from the feasible set leads to a perceived deterioration.We show that in the framework cosidered by Puppe[1996], Chernoff's Axiom and the Generalized Condorcet Axiom (both celebrated in classical rational choice theory) imply our New Quasi Transitivity Axiom, and hence as a consequence of an earlier result,guarantee quasi transitive rationalization.

2.Model Let X be a finite, non-empty universal set. If S is any non-empty subset of X, let [S] denote the set of all non-empty subsets of S. A choice function on X is a function $C : [X] \to [X]$ such that $C(S) \subset S \ \forall \ S \in [X]$. Given a binary relation R on X and $S \in [X]$, let $G(S, R) = \{x \in S / (x, y) \in R \ \forall \ y \in S\}$. This set is called the set of best elements in S with respect to R. Let, $\mathbf{P(R) = \{(x,y) \in R / (y, x) \notin R\}}$ **and** $\mathbf{I(R) = \{(x,y) \in R / (y, x) \in R\}}$. Given a choice function C on X, let $R^C = \{(x,y) \in XxX/x \in C(\{x,y\})\}$ and let $R_C = \cup\{C(S)xS/S \in [X]\}$. **Let** $\mathbf{\Delta=\{(x,x)/x \in X\}}$.

The following result is well known in the literature on rational choice.

Proposition 1 : Given a choice function C on X if there exists a binary relation R on X such that $C(S)= G(S,R) \ \forall \ S \in [X]$, then $R = R^C$.

A binary relation R on X is said to be:

i.) Reflexive if $(x, x) \in R \ \forall \ x \in S$;

ii.) Complete if $\forall \ x, y \in X, x \neq y$ implies $(x, y) \in R$ or $(y, x) \in R$.

iii.) Quasitransitive if $\forall \ x,y,z \in X$, $(x,y) \in P(R)$, $(y,z) \in P(R)$ implies $(x,z) \in P(R)$.

iv) A Quasi-ordering if it satisfies (i), (ii) and (iii).

v) Transitive if $\forall \ x, y, z \in X$, $(x, y) \in R$ & $(y, z) \in R$ implies $(x, z) \in R$;

vi) An Ordering if it satisfies (i), (ii) and (v).

A choice function C is said to satisfy:

a) Chernoff's Axiom (CA) if $\forall \ S,T \in [X]$ with $S \subset T$, $C(T) \cap S \subset C(S)$;

b) Generalized Condorcet (GC) at R if $\forall \ S \in [X]$, $G(S, R^C) \subset C(S)$;

c) Bandopadhyay - Sengupta Acyclicity Axiom (BSAA) at R if $\forall$ S $\in$ [X],[x $\in$ S\C(S) implies that there exist y $\in$ S with (x,y) $\notin R_C$].

Proposition2 : Given a choice function C : [X] $\to$ [X] there exists a binary relation R on X such that C(S) = G(S,R) $\forall$ S $\in$ [X] if and only if at least one of the following two conditions hold:

1) C satisfies CA and GC ;

2) C satisfies BSAA.

The above results are available in Suzumura [1983] and Bandopadhyay and Sengupta [1991].
The reason why we refer to one of the axioms above as an acyclicity axiom is that if $\forall$ S $\in$ [X], C(S) = G(S,R) where R is a binary relation on X, then R must be acyclic in the following sense: there does not exist t ε N and $\{x^i\}_{i=1,\ldots,t}$,all in X such that with $(x^i, x^{i+1}) \in P(R)$ $\forall$ i$\in$ {1,...,t-1} with $(x^t, x^1) \in P(R)$.
A choice function C is said to satisfy the Bandopadhyay Sengupta Quasi Transitivity Axiom (BSQTA) if $\forall$ S $\in$ [X], [x $\in$ S\C(S) implies that there exists y $\in$C(S) with (x,y) $\notin R_C$].

The following result has been established in Bandopadhyay and Sengupta [1991]:

Proposition 3: Given a choice function C : [X] $\to$ [X], there exists a quasi-ordering R on X such that C(S) = G(S,R) $\forall$ S $\in$ [X] only if C satisfies BSQTA.

3.Quasi-Transitive Rationality: A choice function C on X is said to satisfy

d) Outcasting (O) if $\forall$ S, T $\in$ [X], C(T) $\subset$ S $\subset$ T implies C(S) = C(T);

e) Superset Axiom (SUA) if $\forall$ S, T $\in$ [X], C(T) $\subset$ C(S) $\subset$ T implies C(S) = C(T);

f) Jamison and Lau's Quasi Transitivity Axiom (JLQTA) if
$\forall$ S, T $\in$ [X], S $\subset$ T \ C(T) implies C(T \S) = C(T);

g) Sen's Quasi Transitivity Axiom (SQTA) if $\forall$ S,T $\in$ [X], S $\subset$ T, x, y $\in$ C(S), x $\neq$ y implies C(T) $\neq$ {x}.

h) Fishburn's Quasi Transitivity Axiom (FQTA) if $\forall$ S, T $\in$ [X], [S \ C(S)] $\cap$ C(T) $\neq \phi$ implies C(S) \T $\neq \phi$.

Outcasting is generally attributed to Nash [1950]; the Superset Axiom can be found in Suzumura [1983]; Jamison and Lau's Quasi Transitivity Axiom can be found in Jamison and Lau [1973], Sen's Quasi-Transitivity Axiom can be found in Sen [1971]; Fishburn's Quasi-Transitivity Axiom can be found in Fishburn [1975]. The following result can be found in the above mentioned papers and in Aizerman and Aleskerov [1995].

Note: O need not imply CA as the following example reveals: Let X = {x, y, z}, C (X) = {x, y}, C({x,y}) = {x, y}, C({y,z} = {y}, C ({x,z}) = {z}. C satisfies O. However, {x,z} $\subset$ X, x $\in$ C(X) $\cap$ {x, z} but x $\notin$ C ({x,z}). Thus C does not satisfy CA.

Theorem 1: Given a choice function C on X such that C(S) = G(S,R) $\forall$ S $\in$ [X], where R is a binary relation on X, R is a quasi-transitive if and only if any one of the following holds:

i Outcasting;
ii Superset Axiom;
iii Jamsion and Lau's Quasi-Transitivity Axiom;
iv Sen's Quasi-Transitivity Axiom;
v Fishburn's Quasi-Transitivity Axiom.

We now introduce a new quasi-transitivity axiom, similar in spirit to Sen's Quasi-Transitivity Axiom. This Axiom is originally due to Nehring (1997).

New Quasi Transitivity Axiom (NQTA): A choice function C on X is said to satisfy the New Quasi Transitivity Axiom if $\forall$ S $\in$ [X] , [x, y $\in$ S\C(S) implies y$\notin$ C(S\{x})] .

We now introduce the following result:

Theorem 2: Let C be a choice function on X such that C(S) = G (S, R) $\forall$ S $\in$ [X], where R is a binary relation on X. Then R is quasi-transitive if and only if C satisfies NQTA.

Proof: Suppose C(S) = G (S, R) $\forall$ S $\in$ [X], where R is a quasi-ordering on X. Let x, y $\in$ S \ C(S). Since S is finite and R is a quasi-ordering, there exists z $\in$ C(S) such that (z,x) $\in$ P(R). Thus, z $\in$ S \ {y}. Hence, x $\notin$ G (S \ {y}, R) = C(S \ {y}).
Now suppose C(S) = G(S,R) $\forall$ S $\in$ [X] and C satisfies NQTA. Let (x,y) ε P(R), (y,z) ε (P,R). Let S = {x,y,z}. Since C(S) $\neq \varnothing$, we must have C(S) = {x}. Hence (z,x) $\notin$ P(R). If (x,z) $\notin$ P(R), then C ({x,z}) = {x,z}. However, then y, z $\in$ S \ C(S) and z $\in$ C (S \ {y}), contradicts NQTA. Thus, (x,z) $\in$ P(R). This proves the theorem.
♣
Further, by appealing to Proposition 2 and Theorem 2, we may now assert the following :

Theorem 3: Given a choice function C on X, there exists a quasi-ordering R on X such that C(S) = G(S,R) $\forall$ S $\in$ [X] if and only if any one of the following holds :

a) C satisfies CA, GC and NQTA;
b) C satisfies BSAA and NQTA.

4.Complete Logical Independence of CA, GC and NQTA :
Example 1: A choice function which does not satisfy either CA or GC or NQTA: Let X = {x,y,z}, C(X) = {x}, C ({x,y}) = {x,y}, C({y,z}) = {y,z}, C({x,z}) = {z}, C({a}) = {a} $\forall$ a $\in$ X. Since x $\notin$

C({x,z}), C does not satisfy CA. Since $z \notin C(X)$, C does not satisfy GC; since $y, z \in X \setminus C(X)$ and $z \in C(X \setminus \{y\})$, C does not satisfy NQTA. We have here a choice function which does not satisfy BSAA either : $x \notin C(\{x,z\})$ and BSAA implies $(x,z) \notin R_c$. However $z \in X$ and $x \in C(X)$, contradicting BSAA.

Example 2: A choice function which does not satisfy either CA or GC but satisfies NQTA : $X = \{x,y,z\}$, $C(X) = \{x,y\}$, $C(\{x,y\}) = \{x,y\}$, $C(\{y,z\}) = \{y,z\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C does not satisfy CA since $x \notin C(\{x,z\})$; C does not satisfy GC since $z \notin C(X)$. However, C satisfies NQTA. Note C does not satisfy BSAA : $x \notin C(\{x,z\})$ implies by BSAA, $(x,z) \notin R_c$. However $z \in X$ and $x \varepsilon C(X)$, contradicting BSAA.

Example 3: A choice function which does not satisfy either CA or NQTA, but satisfies GC : $X = \{x,y,z\}$, $C(X) = \{x\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C does not satisfy CA, since, $x \notin C(\{x,z\})$; C does not satisfy NQTA, since $y, z \notin C(X)$, but $z \in C(X \setminus \{y\})$. However, C satisfies GC vacuously. Note that C does not satisfy BSAA: $x \notin C(\{x,z\})$ and BSAA imply $(x,z) \notin R_c$ contradicting $z \in X$ and $x \in C(X)$.

Example 4: A choice function which does not satisfy either GC or NQTA but satisfies CA : $X = \{x,y,z\}$, $C(X) = \{x\}$, $C(S) = S\ \forall S \in [X], S \neq X$. C does not satisfy GC since $y \notin C(X)$. C does not satisfy NQTA, since $y, z \in X \setminus C(X)$ but $z \in C(X \setminus \{y\})$. However, C satisfies CA. Note that C does not satisfy BSAA : $y \in X \setminus C(X)$ implies either $(y,x) \notin R_c$ or $(y,z) \notin R_c$ contradicting $y \in C(\{x,y\})$ and $y \varepsilon C(\{y,z\})$.

Example 5: A choice function which does not satisfy CA, but satisfies GC and NQTA : $X = \{x,y,z\}$, $C(X) = X$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C does not satisfy CA, since, $y \notin C(\{x,y\})$. However it satisfies GC and NQTA vacuously. Note C does not satisfy BSAA: $y \in \{x,y\} \setminus C(\{x,y\})$ implies by $(x,y) \notin R_c$ contradicting $y \in X$ and $x \in C(X)$.

Example 6: A choice function which does not satisfy GC, but satisfies CA and NQTA : $X = \{x,y,z\}$, $C(X) = \{x,y\}$, $C(S) = S\ \forall S \in [X], S \neq X$. C does not satisfy GC, since $z \notin C(X)$. C satisfies CA. C satisfies NQTA vacuously. Note C does not satisfy BSAA : $z \varepsilon X \setminus C(X)$ implies by BSAA either $(z,x) \notin R_c$ or $(z,y) \notin R_c$ contradicting $z \in C(\{x,z\})$ and $z \in C(\{y,z\})$.

Example 7: A choice function which does not satisfy NQTA, but satisfies CA and GC : $X = \{x,y,z\}$, $C(X) = \{x\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{x,z\}$. C satisfies CA and GC. But C does not satisfy NQTA: $y, z \in X \setminus C(X)$ and yet $z \in C(X \setminus \{y\})$. Note C satisfies BSAA.

Example 8: A choice function which satisfies CA, GC and NQTA : $X = \{x,y,z,w\}$, $C(X) = \{x,w\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{x\}$, $C(\{x,w\}) = \{x,w\}$, $C(\{z,w\}) = \{z,w\}$, $C(\{y,w\}) = \{y,w\}$, $C(\{x,y,z\}) = \{x\}$, $C(\{x,y,w\}) = \{x,w\}$, $C(\{x,z,w\}) = \{x,w\}$. $C(\{y,z,w\}) = \{y,w\}$. C satisfies CA, GC and NQTA. Note that C satisfies BSAA as well.Let,$R = \{(x,x), (y,y), (z,z), (w,w), (x,y), (y,z), (x,z), (x,w),(w,x), (y,w), (w,y), (z,w), (w,z)\}$. $C(S) = G(S, R)\ \forall S \varepsilon [X]$. R is a quasi-ordering.

However, $(z,w) \in R$ and $(w,x) \in R$. Yet $(z,x) \notin R$. Hence R is not transitive. Thus, R is not an ordering. In view of Proposition 1, we may conclude that there does not exist any ordering on X, such that for every S in [X], C(S) is equal to the set of best in S with respect to the given ordering.

Note: Example 1 above gives an example of a choice function which does not satisfy either BSQTA or NQTA; Examples 2, 5, 6 above gives examples of choice functions which satisfy NQTA but not BSQTA. Thus, in view of Theorem 1 and Theorem 2 we may conclude the following.

Theorem 4 : BSQTA implies NQTA. However, the converse is not true.

5.Relationship of NQTA with other axioms:

It may be interesting to compare the relative strengths of SQTA and NQTA.Towards that goal we have the following result to offer:

Theorem 5: NQTA implies SQTA.However the converse is not true.

Proof : Suppose C satisfies NQTA and towards a contradiction suppose that there exists $A,B \in [X]$ with $B \subset A$ and there exists $x,y \in C(B)$ with $x \neq y$ but $\{x\}=C(A)$.Since this is clearly not possible with B=A, we must have $B \subset\subset A$.Let $A \backslash B=\{x_1,\ldots,x_r\}$,for some positive integer r.Clearly, $y \notin \{x_1,\ldots,x_r\}$. Suppose $C(B\cup\{x_1\})=\{y\}$.Then $x_1, x \in (B\cup\{x_1\}) \setminus C(B\cup\{x_1\})$ and $x \in B$ contradicting NQTA.Thus, $C(B\cup\{x_1\}) \neq \{y\}$.Now suppose $\{y\}\neq C(B\cup\{x_1,\ldots,x_q\})$ for $q<r$.Towards a contradiction suppose, $\{y\}=C(B\cup\{x_1,\ldots,x_{q+1}\})$.
Case1: $y \notin C(B\cup\{x_1,\ldots,x_q\})$. Let $z \in C(B\cup\{x_1,\ldots,x_q\})$.Thus $z \neq y$.Thus, $z, x_{q+1} \in (B\cup\{x_1,\ldots,x_{q+1}\}) \backslash C(B\cup\{x_1,\ldots,x_{q+1}\})$, and $z \in C(B\cup\{x_1,\ldots,x_q\})$, contradicting NQTA.
Case2: $\{y\}\subset\subset C(B\cup\{x_1,\ldots,x_q\})$. Let $z \in C(B\cup\{x_1,\ldots,x_q\})$, with $z \neq y$.Thus, $z, x_{q+1} \in (B\cup\{x_1,\ldots,x_{q+1}\}) \backslash C(B\cup\{x_1,\ldots,x_{q+1}\})$, and $z \in C(B\cup\{x_1,\ldots,x_q\})$, contradicting NQTA.
Hence, $\{y\}\neq C(B\cup\{x_1,\ldots,x_{q+1}\})$.By a simple induction argument,we may conclude that $\{y\}\neq C(A)$. Thus, C satisfies SQTA.
To show that SQTA does not necessarily imply NQTA, let $X=\{x,y,z\}$.Define $C:[X]\rightarrow[X]$ as follows: $C(X)=\{x\}$ and $C(A)=A$ for all $A \in [X]\backslash\{X\}$.C satisfies SQTA vacuously.However, $y,z \in X\backslash C(X)$ and yet, $y \in C(\{x,y\})$, contradicting NQTA.
♣
A choice function C is said to satisfy:
Generalized Axiom of Revealed Preference(GA) if $[y \in A \setminus C(A), C(A) \subset B]$ implies $[y \notin C(B)]$ $\forall$ A, B $\in$ [X] and $y \in X$;

Nehring's Axiom of Revealed Preference (NA) if $y \in A \setminus C(A)$ implies $y \notin C(C(A) \cup \{y\})$;

Aizerman and Malishevski's Axiom (AMA) if $\forall$ A, B $\in$ [X], $[C(A) \subset B \subset A] \rightarrow [C(B) \subset C(A)]$;

GA and NA appear in Nehring [1997] with the latter under the name of ρ_4 ; AMA originates in the work of Aizerman and Malishevski [1981]. This axiom has been used in Nehring and Puppe [1999], and hence the main result reported here, has obvious implications in that paper as well.

Theorem 6:- AMA $\leftrightarrow$ NQTA

Proof: The fact that AMA implies NQTA is obvious. Hence let us prove the converse and that too by induction. Thus suppose C is a choice function which satisfies NQTA. Let A, B $\in$ [X], $C(A) \subset B \subset A$, and x be an arbitrary element of $A \setminus C(A)$. We prove our result by backward induction on the cardinality of B.
Let $B = A \setminus \{y\}$ for some $y \in A \setminus C(A)$. By NQTA, $x \notin C(B)$. Since x is arbitrary, $C(B) \subset C(A)$ whenever $B = A \setminus \{y\}$ and $y \in A \setminus C(A)$.
Now suppose for any $y_1, \ldots, y_r \in A \setminus C(A)$, if $B = A \setminus \{y_1, \ldots, y_r\}$, then $C(B) \subset C(A)$.
Let $y_{r+1} \in A \setminus C(A)$, $y_{r+1} \notin \{y_1, \ldots, y_r\}$.
Let $\overline{B} = A \setminus \{y_1, \ldots, y_r\}$ and thus $B = \overline{B} \setminus \{y_{r+1}\}$
By NQTA, $C(B) \subset C(\overline{B})$. However, by the induction hypothesis, $C(\overline{B}) \subset C(A)$. Hence, $C(B) \subset C(A)$.
Since the result has been proved for r = 1 and has now been shown to be true for r+ 1 if it assumed true for r, it is therefore true in general.
♣

Theorem 7:- AMA & CA $\leftrightarrow$ GA.

Proof: Let C be a choice function which satisfies AMA and CA. Let A, B $\in$ [X] and let $y \in A \setminus C(A)$ with $C(A) \subset B$.
Consider $A \cap B$. Clearly $C(A) \subset A \cap B \subset A$. By AMA, $C(A \cap B) \subset C(A)$.
By CA, $A \cap B \subset B$ implies $C(B) \cap (A \cap B) \subset C(A \cap B)$.
Thus $C(B) \cap A \subset C(A \cap B) \subset C(A)$.
Thus $y \notin C(B)$.
Thus C satisfies GA.
Conversely, let C satisfy GA. Then it obviously does satisfy AMA. To show that it satisfies CA, let A, B $\in$ [X] with $A \subset B$. Let $x \in C(B) \cap A$. If $x \notin C(A)$, then since $C(A) \subset B$, by GA, $x \notin C(B)$ which is a contradiction. Thus, $x \in C(A)$. Thus $C(B) \cap A \subset C(A)$. Thus C satisfies CA.
♣

Example 9: AMA ($\leftrightarrow$ NQTA) does not necessarily imply GA: Let $X = \{x, y, z\}$, $C(X) = \{x, y\}$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C satisfies AMA (and NQTA). However, $y \in \{x, y\} \setminus C(\{x, y\})$, $C(X) \subset \{x, y\}$ and yet $y \in C(X)$. Thus C does not satisfy GA.

Example 10: CA does not necessarily imply GA: Let $X = \{x, y, z\}$, $C(X) = \{x\}$, $C(A) = A\ \forall A \in [X]$, $A \neq X$. C satisfies CA. However, $y \in X \setminus C(X)$, $C(X) \subset \{x, y\}$ and yet $y \in C(\{x, y\})$. Thus C does not satisfy GA.

Theorem 8: (a) CA & AMA implies O; O implies AMA;
(b) AMA need not imply O.

Proof: (a)is easy to establish;(b)Let $X = \{x, y, z\}$, $C(X) = \{x, y\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\} = \{y\}$, $C(\{x,z\}) = \{z\}$. C satisfies AMA. However, $C(X) = \{x, y\} \subset \{x, y\} \subset X$, but $C(\{x, y\}) \neq C(X)$. Thus C does not satisfy OA.

♣

We may thus state the following theorem:

Theorem 9: Given a choice function C on X, there exists a quasi-ordering R on X such that $C(S) = G(S,R)\ \forall S \in [X]$ if and only if any one of the following holds :

a) C satisfies CA, GC and AMA;
b) C satisfies GC and GA.

The following observation is worth noting:

Observation: AMA implies NA. However, the converse need not be true.

Proof: Let C satisfy NA and let $A \in [X]$, $y \in X$ with $y \in A \setminus C(A)$. Thus $C(A) \subset C(A) \cup \{y\} \subset A$. By AMA, $C(C(A) \cup \{y\}) \subset C(A)$. Thus, $y \notin C(C(A) \cup \{y\})$. Thus C satisfies NA.

We show that the converse need not be true by means of an example. Let $X = \{x, y, z, w\}$. Let $C(X) = \{x\}$; $C(A) = A\ \forall A \in [X]$ with three elements; and for all $A \in [X]$ with one or two elements, $C(A) = \{x\}$ if $x \in A$ and $C(A) = A$ otherwise. Now $y, z \in X \setminus C(X)$ and yet $y \in C(X \setminus \{z\})$. Thus C does not satisfy NQTA, which has been shown in Theorem 1 above to be equivalent to AMA. Yet C satisfies NA.

♣

Our primary reason for invoking and emphasising NQTA is because of the significant role it plays in obtaining a neat characterization of quasitransitively rationalizable choice functions generated by a monotonic preference for freedom, as we shall observe later on.

The analysis of quasi-transitive rational choice acquires added relevance in view of the close relationship that exists between quasi-transitive binary relations and the class of comparison functions defined in Dutta and Laslier [1999].A function $g: X \times X \rightarrow \Re$, where $\Re$ is the set of real numbers, is called a comparison function if $\forall x, y \in X$, $g(x,y) = - g(y,x)$.This obviously implies that $g(x,x) = 0\ \forall x \in X$.

Given $x, y \in X$, we say that x covers y via g if $g(x,y)>0$ and $g(x,z) \geq g(y,z)\ \forall z \in X$ .Let $P(g) = \{(x,y)/ x$ covers y via $g\}$and let $R(g)=\{(x,y)/ (y,x) \notin P(g)\}$.Clearly,$P(R(g)) = P(g)$, and it is easy to verify that $R(g)$ is reflexive,complete and quasi-transitive.The following observation which is proved in the appendix implies that a choice function C is rationalizable by a reflexive,complete and quasi-transitive binary relation R if and only if, for some comparison function g, C(S) = G(S,R(g)) $\forall$ S $\in$ [X]:

Observation: A binary relation R is reflexive,complete and quasi-transitive if and only if R=R(g) for some comparison function g.

6.Monotonic Preference for Freedom :A binary relation $\Im$ on [X] is any non-empty subset of [X]x[X]. Let **Γ={(S,S)/S$\in$ [X]}**.$\Im$ is said to be reflexive if $\Gamma \subset \Im$. $\Im$ is said to be transitive if (S,T),(T,U) $\in\Im$ implies (S,U) $\in\Im$. Let, **M={(S,T)/T$\subset$S}**. Note $\Gamma\subset$ M .that $\Im$ is said to be Monotonic with respect to Set Inclusion (MSI)if M$\subset\Im$.Given $\Im$ let P($\Im$)={(S,T) $\in\Im$/(T,U)$\notin\Im$} and I($\Im$)={(S,T) $\in\Im$/(T,S) $\in\Im$}.$\Im$ is said to satisfy Preference for Freedom of Choice(PFC)if $\forall$ S$\in$[X] which has atleast two elements,there exists x$\in$S with (S,S\{x})$\in$ P($\Im$).A binary relation $\Im$on [X] which is reflexive,transitive and satisfies PFC is said to be a Preference for Freedom (PF).If in addition it satisfies MIS, it is said to be a Monotonic Preference for Freedom (MPF).

Let $\Im$ be a MPF.Define $E_\Im$:[X]$\to$[X] as follows:

$E_\Im$(S)=S ,if S has exactly one element;

= {x $\in$X/(S,S\{x})$\in$ P($\Im$)},otherwise.

It is easy to see that $E_\Im$ is well defined by virtue of PFC.

Let $R_\Im$ ={(x,y)/x$\in$ $E_\Im$({x,y}).

Observation:Given an MPF $\Im$ if there exists a binary relation R on X such that $E_\Im$(S)=G(S,R) $\forall$ S$\in$[X],then R= $R_\Im$.

A PF $\Im$ is said to satisfy Independence with respect to Non-essential Alternatives (INA) if ($E_\Im$(S),S) $\in\Im$, for all S$\in$[X].

Theorem 10: Let $\Im$ be an MPF. $\Im$ satisfies INA if and only if $E_\Im$ satisfies NQTA.

Proof: Suppose $\Im$ satisfies INA and towards a contradiction suppose that there exists S$\in$[X] with x,y $\in$S\ $E_\Im$(S) and y $\in$ $E_\Im$(S\{x}).Hence x$\neq$y.By INA and MIS, ($E_\Im$(S),S) $\in$I($\Im$).Now, y $\in$ $E_\Im$(S\{x}) implies (S\{x},S\{x,y})$\in$P($\Im$) and by MSI ,(S,S\{x})$\in\Im$.Since x,y$\in$S\ $E_\Im$(S), $E_\Im$(S) $\subset$ S\{x,y}.By MSI, (S\{x,y}, $E_\Im$(S)) $\in\Im$, so that by transitivity of $\Im$ (S, $E_\Im$(S)) $\in$P($\Im$) contradicting INA.Hence $E_\Im$ satisfies NQTA.

Now suppose $E_\Im$ satisfies NQTA and let S$\in$[X].If $E_\Im$(S)=S, then clearly ($E_\Im$(S),S) $\in\Im$, by reflexivity of $\Im$.Hence, suppose $E_\Im$(S) $\subset\subset$ S.Let S\ $E_\Im$(S)={x_1,...,x_r},for some positive integer r.By MSI and definition of $E_\Im$, (S,S\{ y})$\in$I($\Im$) for y $\in${x_1,...,x_r}. Hence if r=1, (S, $E_\Im$(S)) $\in$I($\Im$).Thus,suppose r >1. Suppose (S\{ y_1,...,y_q}, S\{ y_1,...,y_{q+1}})$\in$I($\Im$) ,$\forall$ { y_1,...,y_{q+1}} $\subset$ {x_1,...,x_r} and for q=1,...,s < r-1. Consider, S\{ y_1,...,y_{s+1}}.Now, y_{s+1} ,$y_{s+2}$$\in$ (S\{ y_1,...,y_s})\ $E_\Im$ (S\{ y_1,...,y_s}).By,NQTA, $y_{s+2}$$\in$ (S\{ y_1,...,y_s ,y_{s+1} })\ $E_\Im$ (S\{ y_1,...,y_s ,y_{s+1} }).Thus, (S\{ y_1,...,y_{s+1}}, S\{ y_1,...,y_{s+2}})$\in$I($\Im$).By a standard induction argument and transitivity of $\Im$, ($E_\Im$(S),S) $\in\Im$. Thus $\Im$ satisfies INA.

♣

We have already seen in Theorem 5,that NQTA implies SQTA and that the converse is not true.

Let X={x,y,z} and let $\mathfrak{I}$ = M ∪ {(S,T) ∈[X]x[X] /x ∈ S and S has atleast two elements}.It is easily checked that $\mathfrak{I}$ is an MPF and $E_{\mathfrak{I}} \equiv C$, where C is the choice function in Theorem 5 which satisfied SQTA but did not satisfy NQTA.Hence, we may assert:

Proposition 4: There exists an MPF $\mathfrak{I}$ such that $E_{\mathfrak{I}}$ satisfies SQTA but does not satisfy NQTA.

Theorem 11: Let $\mathfrak{I}$ be an MPF such that $E_{\mathfrak{I}}$ satisfies CA and GC.Then $E_{\mathfrak{I}}$ satisfies NQTA.

Proof : Let $\mathfrak{I}$ be an MPF such that $E_{\mathfrak{I}}$ satisfies CA and GC and towards a contradiction suppose that there exists S ∈[X] with x,y ∈S\\$E_{\mathfrak{I}}$(S) and yet y ∈$E_{\mathfrak{I}}$(S\\{x}).Hence since $\mathfrak{I}$ is an MPF:
(1) (S,S\\{x})∈ I($\mathfrak{I}$);
(2) (S,S\\{y})∈ I($\mathfrak{I}$);
(3) (S\\{x},S\\{x,y})∈ P($\mathfrak{I}$).

Thus by transitivity of $\mathfrak{I}$, we get (S\\{x},S\\{y})∈ I($\mathfrak{I}$) which leads to (S\\{x},S\\{x,y})∈ P($\mathfrak{I}$) , once again by the transitivity of $\mathfrak{I}$.Hence, by the definition of $E_{\mathfrak{I}}$, x ∈$E_{\mathfrak{I}}$(S\\{y}).By CA, x ∈$E_{\mathfrak{I}}$({x,z}) ∀ z ∈ S\\{y} and y ∈$E_{\mathfrak{I}}$({y,z}) ∀ y ∈ S\\{y}.Since, x,y ∈S\\$E_{\mathfrak{I}}$(S) by GC, $E_{\mathfrak{I}}$({x,y}) = ϕ, contradicting the fact that the range of $E_{\mathfrak{I}}$ does not contain the empty set.Hence $E_{\mathfrak{I}}$ satisfies NQTA.
♣

Corollary to Theorem 11: Let $\mathfrak{I}$ be an MPF such that $E_{\mathfrak{I}}$ satisfies CA and GC. Then $E_{\mathfrak{I}}$(S) = G(S, $R_{\mathfrak{I}}$) for all S∈[X], and $R_{\mathfrak{I}}$ is a quasi-order.

Proof: Follows easily from Theorems 3 and 11.
♣

Example 11: An MPF $\mathfrak{I}$ such that $E_{\mathfrak{I}}$ satisfies CA and NQTA but does not satisfyy GC : Let X={x,y,z} and let $\mathfrak{I}$ = M ∪ {({x,y},S)/S∈[X]}. $\mathfrak{I}$ is an MPF.Now, $E_{\mathfrak{I}}$(X) ={x,y}, and $E_{\mathfrak{I}}$(S) =S for all S∈[X] \\{X}.Clearly $E_{\mathfrak{I}}$ satisfies CA and NQTA.However z ∈ $E_{\mathfrak{I}}$({z,a}) ∀ a ∈ X, and yet z ∈ X\\ $E_{\mathfrak{I}}$(X) contradicting GC.

Example 12: An MPF $\mathfrak{I}$ such that $E_{\mathfrak{I}}$ satisfies GC and NQTA but does not satisfyy CA : Let X={x,y,z} and let $\mathfrak{I}$ = M ∪ {({x},S)/S∈[X], and S has atmost two elements}∪{({y},S)/ S∈[X] and x∉ S}. $\mathfrak{I}$ is an MPF. Now, $E_{\mathfrak{I}}$(X) =X, $E_{\mathfrak{I}}$({x,y})= {x}= $E_{\mathfrak{I}}$({x,z}), $E_{\mathfrak{I}}$({y,z})={y} and $E_{\mathfrak{I}}$(S)=S otherwise.Clearly $E_{\mathfrak{I}}$ satisfies GC and NQTA.However,z ∈ $E_{\mathfrak{I}}$(X)∩ {x,z}, but z ∈{x,z}\\ $E_{\mathfrak{I}}$({x,z}) contradicting CA.

Example 13: An MPF $\mathfrak{I}$ such that $E_{\mathfrak{I}}$ satisfies CA but does not satisfyy either GC or NQTA : Let X={x,y,z} and let $\mathfrak{I}$ = M ∪ { ({x, y}, S)/ S ∈ [X] } ∪ { ({x, z}, S)/ S ∈ [X] }. $\mathfrak{I}$ is an MPF. Now, $E_{\mathfrak{I}}$(X) = {x}, $E_{\mathfrak{I}}$(S) = S otherwise. Clearly $E_{\mathfrak{I}}$ satisfies CA and NQTA. However, y ∈ $E_{\mathfrak{I}}$ ({x, y}) ∩ $E_{\mathfrak{I}}$ ({y, z}), but y ∈ X\\$E_{\mathfrak{I}}$(X), contradicting GC.

Example 14: An MPF $\Im$ such that $E_\Im$ satisfies GC but does not satisfyy either CA or NQTA : Let X={x,y,z,w} and let $\Im$ = M ∪ { ({x, y,z}, S)/ S ∈ [X] } ∪ { ({x, y,w}, S)/ S ∈ [X] }∪ {({x}, {x,y}), ({x}, {y}). $\Im$ is an MPF. Now, $E_\Im$(X) = {x,y}, $E_\Im$({x,y}) = {x}, $E_\Im$(S) = S otherwise. Clearly $E_\Im$ satisfies GC. However, w,z ∈ X\$E_\Im$ (X) and w ∈ $E_\Im$ ({x,y, w}), contradicting NQTA. Further, y∈$E_\Im$(X) ∩ {x,y}, and yet y ∈ {x, y}\ $E_\Im$ ({x, y}), thus contradicting CA.

Proposition 5: Suppose X has three or less elements. Then for any MPF $\Im$, [$E_\Im$ satisfies GC] implies [$E_\Im$ satisfies NQTA].

Proof: If X has one or two elements then NQTA is satisfied vacuously.The same is true if X has three elements and $E_\Im$(X) has two elements.Hence assume X has three elements and without loss of generality suppose $E_\Im$(X) = {x}.Thus (X,{x,y}) and (X,{x,z}) ∈ I($\Im$) and by transitivity of $\Im$ ({x,y},{x,z}) ∈ I($\Im$) as well. Now a violation of NQTA occurs if (without generality) y ∈ $E_\Im$ ({x, y}) . Thus, ({x,y},{x}) ∈ P($\Im$) and consequently (since ({x,y},{x,z}) ∈ I($\Im$)),({x,z},{x}) ∈ P($\Im$). However, then z ∈ $E_\Im$ ({x, z}). Since $E_\Im$({y,z}) ∩ {y,z}≠ ϕ, {y,z}⊂ X\ $E_\Im$(X), leads to a contradiction of GC, thereby proving the proposition.

♣

Example 15: An MPF $\Im$ such that $E_\Im$ satisfies NQTA but does not satisfy either CA or GC : Let X={x,y,z} and let $\Im$ = M ∪ { ({y, z}, S)/ S ∈ [X] } ∪ { ({x}, S)/ S ∈ {{x,y},{x,z},{y},{z}}}. $\Im$ is an MPF. Now, $E_\Im$(X) = {y,z}, $E_\Im$({x,y})= {x}= $E_\Im$({x,z}), $E_\Im$({y,z})={y,z} and $E_\Im$(S) = S otherwise. Clearly $E_\Im$ NQTA. However, x ∈ $E_\Im$ ({x, y}) ∩ $E_\Im$ ({x, z}), but x ∈ X\$E_\Im$(X), contradicting GC. Further, z ∈ $E_\Im$(X)∩ {x,z}, but z ∈{x,z}\ $E_\Im$({x,z}) contradicting CA.

Example 16: An MPF $\Im$ such that $E_\Im$ satisfies both CA and GC and hence NQTA : Let $\Im$ ={(S,T) ∈ [X]x[X] / cardinality of S is not less than the cardinality of T}.This is the MPF due to Pattanaik and Xu [1990]. The corresponding $E_\Im$ is easily seen to satisfy CA,GC and NQTA.The interesting thing about this MPF is that $R_\Im$ =XxX.

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Appendix

Theorem: A binary relation R is reflexive,complete and quasi-transitive if and only if R=R(g) for some comparison function g.

Proof: Let g be a comparison function with R(g) and P(g) being as defined in Section 4. It is easy to see that R (g) is reflexive an complete. Hence let us show that P(g) is transitive. Let (x, y), (y, z) $\in$ P(g). Thus :

(a) $g(x, y) > 0, g(y, z) > 0$

(b) $g(x, w) \geq g(y, w), g(y, w) \geq g(z, w) \ \forall \ w \in X$.

Thus $g(x, z) \geq g(y, z) > 0$

and $g(x, w) \geq g(z, w) \ \forall \ w \in X$.

Thus $(x, z) \in P(g)$.

Thus P(g) is transitive.

Now let R be a reflexive, complete and quasi-transitive binary relation. Given $x \in X$, let

$P(x) = \{y \in X / (y, x) \in P(R)\}$ and $I(x) = \{y \in X / (y, x) \in I(R)\}$.
Let $R(x) = P(x) \cup I(x)$. Clearly $R(x) \neq \phi$, since $x \in R(x)$.
Let $A^1(x) = G(R, R(x))$.Having defined $A^i(x)$ for $i = 1, ..., k$, let

$$A^{k+1}(x) = G(R, R(x) \; (\bigcup_{i=1}^{k} A^i(x))).$$

Since R(x) is finite, there exists a positive integer K : $R(x) = \bigcup_{i=1}^{K} A^i(x)$. Observe, $i \neq j$, $j \in \{1, ..., k\}$ implies $A^i(x) \cap A^j(x) = \phi$. Let $g(y, x) = K - i$ if $y \in A^i(x) \subset R(x)$.
Clearly $g(x, x) = 0$. Since R is complete, $y \notin R(x)$ implies $x \in P(y)$. Let, $g(y, x) = - g(x, y)$ if $y \notin R(x)$.
Thus $g : X \times X \to \Re$, is indeed a comparison function. Let $(x, y) \in P(R)$. Thus, $x \in P(y)$ and hence $g(x, y) > 0$. Let $z \in X \setminus \{x, y\}$.
Case 1 :- $y \in R(z)$, $x \notin R(z)$.
Thus $(y, z) \in R$ and $(z, x) \in P(R)$.
If $(y, z) \in P(R)$, then transitivity of P(R) implies $(y, x) \in P(R)$ contradicting $(x, y) \in P(R)$. Thus $(y, z) \in I(R)$. But then $(z, x) \in P(R)$ & $(x, y) \in P(R)$ implies by the transitivity of P(R) that $(z, y) \in P(R)$ contradicting$(y, z) \in I(R)$. Hence Case 1 is not possible.
Case 2 :- $y \in R(z)$ and $x \in R(z)$. In this case, since$(x, y) \in P(R)$, the method of construction of g shows that $g(x, z) > g(y, z)$, since y cannot be chosen while x is still available.
Case 3:- $y \notin R(z)$ and $x \in R(z)$. in this case, $g(x, z) > g(y, z)$ by the definition of g.
Case 4 :- $y \notin R(z)$ and $x \notin R(z)$. Thus $z \in P(x)$ and $z \in P(y)$. Now $(x, y) \in P(R)$ implies, $x \in P(y)$.
Since$(z, x) \in P(R)$ and $(x, y) \in P(R)$, if $(z, w) \in P(R)$ & $(w, x) \in P(R)$ for some $w \in X$, then $(w, y) \in P(R)$ as well. Thus $g(z, x) < g(z, y)$. Hence $g(y, z) > g(x, z)$,since $g(y, z) = g(z, y)$ and $g(x, z) = g(z, x)$.
Hence $(x, y) \in P(g)$.Thus $P(R) \subset P(g)$.
Now suppose $(x, y) \in P(g)$. Thus $g(x, y) > 0$.
Hence $x \in P(y)$. Thus $(x, y) \in P(R)$. Thus, $P(g) \subset P(R)$. This combined with $P(R) \subset P(g)$ yields $P(g) = P(R)$.
Now suppose, $(x, y) \in I(R)$. Thus $x \in I(y)$ and $y \in I(x)$. Thus $g(x, y) = g(y, x) = 0$. Thus, $(x, y) \notin P(g)$ and $(y, x) \notin P(g)$. Thus $(x, y) \in I(g)$. Thus $I(R) \subset I(g)$. Now suppose $(x, y) \in I(g)$. If $(x, y) \in P(R)$, then since $P(R) = P(g)$, $(x, y) \in P(g)$ which is not possible. Thus, $(x, y) \notin P(R)$. Similarly, $(y, x) \notin P(R)$. By completeness and reflexivity of R, $(x, y) \in I(R)$. Thus $I(g) \subset I(R)$. This combined with $I(R) \subset I(g)$ gives us $I(R) = I(g)$.Since R and R(g) are reflexive and complete, we get $R = R(g)$.