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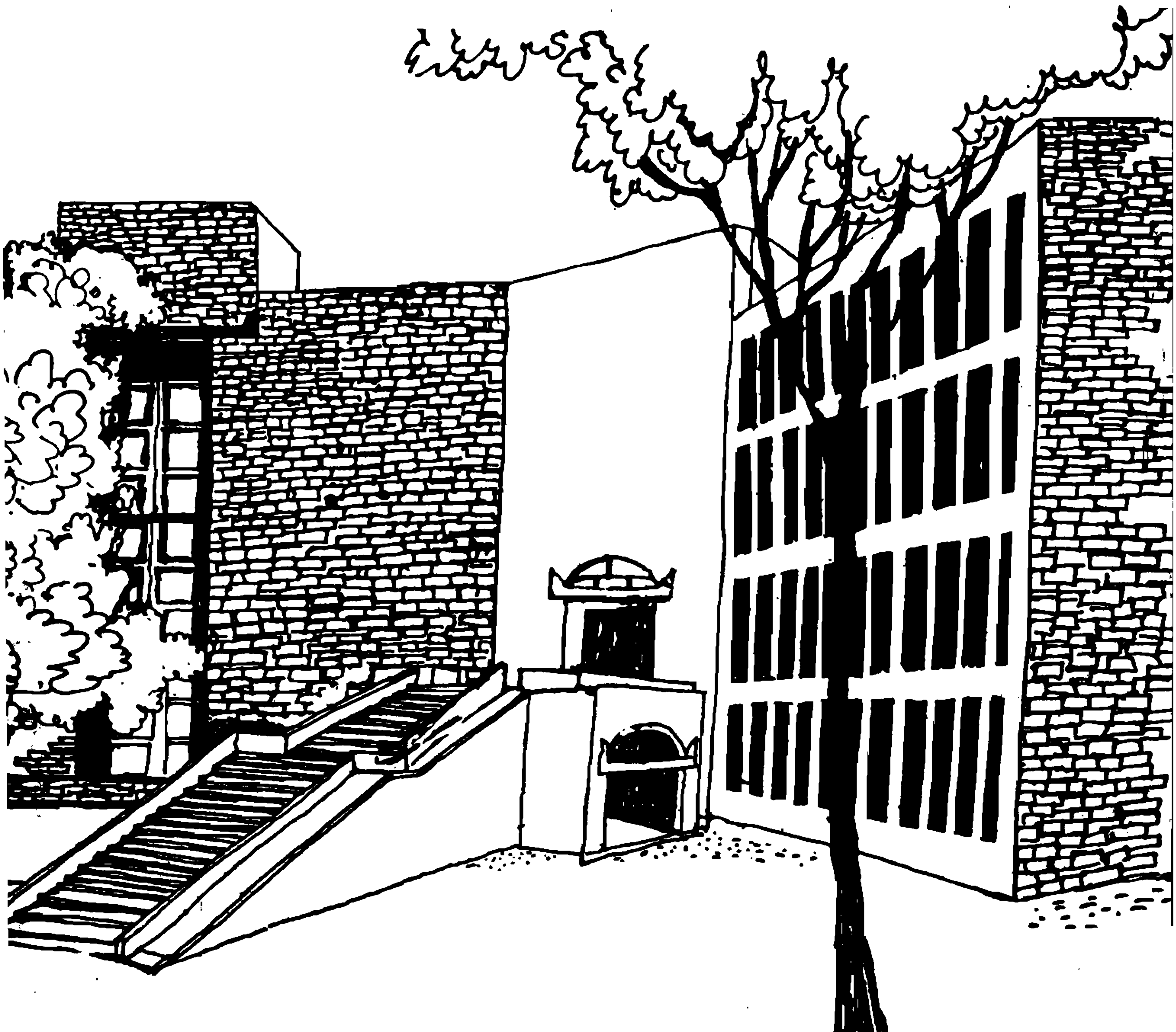

PRESERVATION OF PROXIMITY AND MERGING FUNCTIONS

By

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Abstract

In this paper we are interested in a property due to Baigent (1987) called proximity preserving.In the conventional model of voting theory it was proposed by Baigent that aggregation procedures should be proximity preserving in the sense that given three preference profiles if the second is closer to the first than the third according to an additively separable metric then the second social ranking should also be closer to the first social ranking compared to the third social ranking.In this framework distance between profiles is measured as the sum of distances between the preferences of each agent. In this paper we assume a metric on the space of alternatives and thereby generate an entire family of metrics on ballot profiles. An interesting special case of our family of metrics is the distance between two profiles measured as the sum of the distances between the candidates of each agent on the two ballot profiles. In this framework we obtain the result that there is no merging function which satisfies anonymity and the proximity preservation property. A particular case of a merging function is when the universal set of alternatives is finite and the elected outcome is required to be a candidate who must have received atleast one vote.We call such merging functions vote aggregators. It therefore follows as a consequence of the above result that there is no vote aggregator which satisfies anonymity and the proximity preservation property.Two similar results, one about social welfare functions and the other about social decision functions can be found in Baigent (1987).However, not only is the context of our analysis different, but the method of proof bears little resemblance to the ones available in the work just cited.

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Introduction : - The conventional model of voting is one where a finite set of agents announce there rankings of a finite set of alternatives and then a social welfare function aggregates these announcements into a social ranking of alternatives.The seminal work of Arrow dealt with the observation that if the social ranking is such that (i)given two alternatives if one is preferred ove the other unanimously then the social ranking also ranks the alternatives similarly; and (ii) the relative social ranking between two alternatves depends only on the relative individual rankings between these two alternatives, then the social ranking is nothing but the ranking of a single individual i.e. the dictator.

In recent times the more realistic possibility of each individual in a society casting a ballot and a voting operator aggregating the ballots into elected outcomes has been modelled in Lahiri (1999,2000).In Aczel and Roberts (1989), one is introduced to the idea of a merging function which aggregates ballots which are singletons into a singleton outcome.This is definitely a more realistic model of democratic exercises as we see it . The merging function is required to satisfy the rather innocuous assumption called unanimity;i.e. if every one votes for the same candidate then that candidate is elected.Quesada (2000) does a detailed analysis of the manipulability properties of merging functions, i.e. the existence of a voter who can affect the outcome of the merging function unilaterally,irrespective of who all the others vote for.

In this paper we are interested in a property due to Baigent (1987) called proximity preserving.In the conventional model of voting theory it was proposed by Baigent that aggregation procedures should be proximity preserving in the sense that given three preference profiles if the second is closer to the first than the third according to an additively separable metric then the second soci ranking should also be closer to the first social ranking compared to the third social ranking.In this framework distance between profiles is measured as the sum of distances between the preferences of each agent. In this paper we assume a metric on the space of alternatives and thereby generate an entire family of metrics on ballot profiles. An interesting special case of our family of metrics is the distance between two profiles measured as the sum of the distances between the candidates of each agent on the two ballot profiles. In this framework we obtain the result that there is no merging function which satisfies anonymity and the proximity preservation property. A particular case of a merging function is when the universal set of alternatives is finite and the elected outcome is required to be a candidate who must have received atleast one vote.We call such merging functions vote aggregators. It therefore follows as a consequence of the above result that there is no vote aggregator which satisfies anonymity and the proximity preservation property.Two similar results, one about social welfare functions and the other about social decision functions can be found in Baigent (1987).However, not only is the context of our analysis different, but the method of proof bears little resemblance to the ones available in the work just cited.

The Model : - Let $I = \{1,2,\ldots,n\}$, $(1<n<\infty)$ denote a set of individualts. Let X denote a non-empty set of alternatives containing atleast two distinct alternatives. Let, X^I denote the set of all functio from I to X. If x belongs to X, then x(i) is denoted x_i. For x in X, let range (x) = $\{x_i / x_i = x(i)$ and $i \in I\}$. Let $\Delta = \{x \in X^I / x_i = x_j\}$ be the diagonal of X^I. A merging function on X, is a function $f : X^I \rightarrow X$ such that if $x \in \Delta$ then $f(x) = p = x_i \forall i \in I$. This latter property is often referred to as unanimity.
A merging function f on X is said to be a vote aggregator, if (i) X is finite; (ii) $\forall x \in X^I$: $f(x) \in$ range (x).

Let δ be any metric on X and let "r" be any positive real number. Define a (δ, r) induced metric d on X^I as follows : $\forall x,y \in X^I$: $d_\delta^r(x,y) = \left\{\sum_{i \in j}^{\delta}[\delta(x_i,y_i)]^r\right\}^{\frac{1}{r}}$.

Preservation of Proximity :- A merging function $f : X^I \rightarrow X$ is anonymous if whenever $x,y \in X^I$ and $j,k \in I$: $[x_i=y_i \; \forall i \in I \setminus \{j,k\}, x_j=y_k, x_k=y_j]$ implies $f(x) = f(y)$. It is said to preserve proximity, if $\forall x,y,z \in X^I$: $\left[d_\delta^r(x,y) < d_\delta^r(x,z)\right]$ implies $[\delta(f(x),f(y)) \leq \delta(f(x),f(z))]$.

Theorem 1 :- There is no merging function which is anonymous and preserves proximity.

Proof :- Suppose f satisfies anonymity .
Case 1 ;- n is an even number.
Let $p,q \in X$ with $p \neq q$.
Let $x \in X^I$ with $x_i = p \forall i \in I$.
$y \in X^I$ with $y_i = q \forall i \in I$
$z \in X^I$ with $z_i = p$ if $i \in I$ and i is odd
$= q$ if $i \in I$ and i is even
$w \in X^I$ with $w_i = q$ if $i \in I$ and i is odd
$= p$ if $i \in I$ and i is even.
By anonymity, $f(z) = f(w)$ and $f(x)=p \neq q=f(y)$.

Now $d_\delta^r(x,z) = d_\delta^r\{x,w) = d_\delta^r(y,z) = d_\delta^r(y,w) = \left(\frac{n}{2}\right)^{\frac{1}{r}}\delta(p,q)$, and $d_\delta^r(x,y) = (n)^{\frac{1}{r}}\delta(p,q)$. If $f(x) \neq p$, then $\delta(f(z),f(x)) > 0$, but $\delta(f(z),f(w))=0$.

Since $d_\delta^r(z,w) = (n)^{\frac{1}{r}}\delta(p,q) > \left(\frac{n}{2}\right)^{\frac{1}{r}}\delta(p,q) = d_\delta^r(z,x)$, f does not preserve proximity.

If f(Q)=p, then $\delta(f(Q),f(P))>0$, but $\delta(f(Q),f(M))=0$. Since
$d_\delta^r(Q,M) = (n)^{\frac{1}{r}}\delta(p,q) > \left(\frac{n}{2}\right)^{\frac{1}{r}}\delta(p,q) = d_\delta^r(Q,P)$, f does not preserve proximity.

Case 2 :- n is an odd number greater than or equal to 5.

Then $n-1 > \left(\frac{n+1}{2}\right) > \left(\frac{n-1}{2}\right)$. As in case 1, let p,q $\in$ X with $p \neq q$. Let x,y,z be as in case 1. Let $w \in X^I$ with w_i = q if $i \in I$ and i is odd $i \neq n$

= p if $i \in I$ and i is even or i=n.

Now,

$d_\delta^r(x,z) = \left(\frac{n-1}{2}\right)^{\frac{1}{r}} \delta(p,q)$ and $d_\delta^r(y,z) = \left(\frac{n+1}{2}\right)^{\frac{1}{r}} \delta(p,q)$.

Further, $d_\delta^r(z,w) = (n-1)^{\frac{1}{r}} \delta(p,q)$. Thus $d_\delta^r(z,w) > \max\{d_\delta^r(x,z), d_\delta^r(y,z)\}$.

By anonimity, f(z) = f(w) and f(x) = p, f(y)=q.

If $f(z) \neq p$, then $\delta(f(z),f(x)) > 0 = \delta(f(z),f(w))$ violates preservation of proximity.

If f(z)=p, then $\delta(f(z),f(y))>0 = \delta(f(z),f(w))$ violates preservation of proximity.

Case 3 :- n = 3.

Let p,q,x,y,z,w be as in case 2.If $f(z)\neq p$,then a violation of proximity preservation occurs exactly as in case 2. Hence suppose f(z) = p and towards a contradiction suppose f preserves proximity. By anonymity f(z') = p whenever, $z' \in X^I$ and $\#\{i \in I / z'_i = p\} \geq 2$. Let $\bar{z} \in X^I$,with $\bar{z}_1 = q$, $\bar{z}_2 = p$, $\bar{z}_3 = q$. Now, $d_\delta^r(z,w) = 2^{\frac{1}{r}} \delta(p,q)$ and $d_\delta^r(\bar{z},w) = \delta(p,q)$. Since f is supposed to preserve proximity, f(z) = f(w) implies $f(\bar{z}) = f(w) = f(z) = p$. Let $\bar{w} \in X^I$,with $\bar{w}_1 = p$, $\bar{w}_2 = q$, $\bar{w}_3 = q$. By anonymity, $f(\bar{w}) = f(\bar{z}) = p \neq q$. Now, $d_\delta^r(\bar{z},\bar{w}) = 2^{\frac{1}{r}} \delta(p,q) > \delta(p,q) = d_\delta^r(\bar{z},y)$. Since preserves proximity and $f(\bar{w}) = f(\bar{z})$ = p, f(y) must also be p. However f(y) = q $\neq$p. This contradiction establishes that f does not preserve proximity.□

It follows as a consequence of Theorem 1, that there is no vote aggregator which satisfies anonymity and preserves proximity.

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