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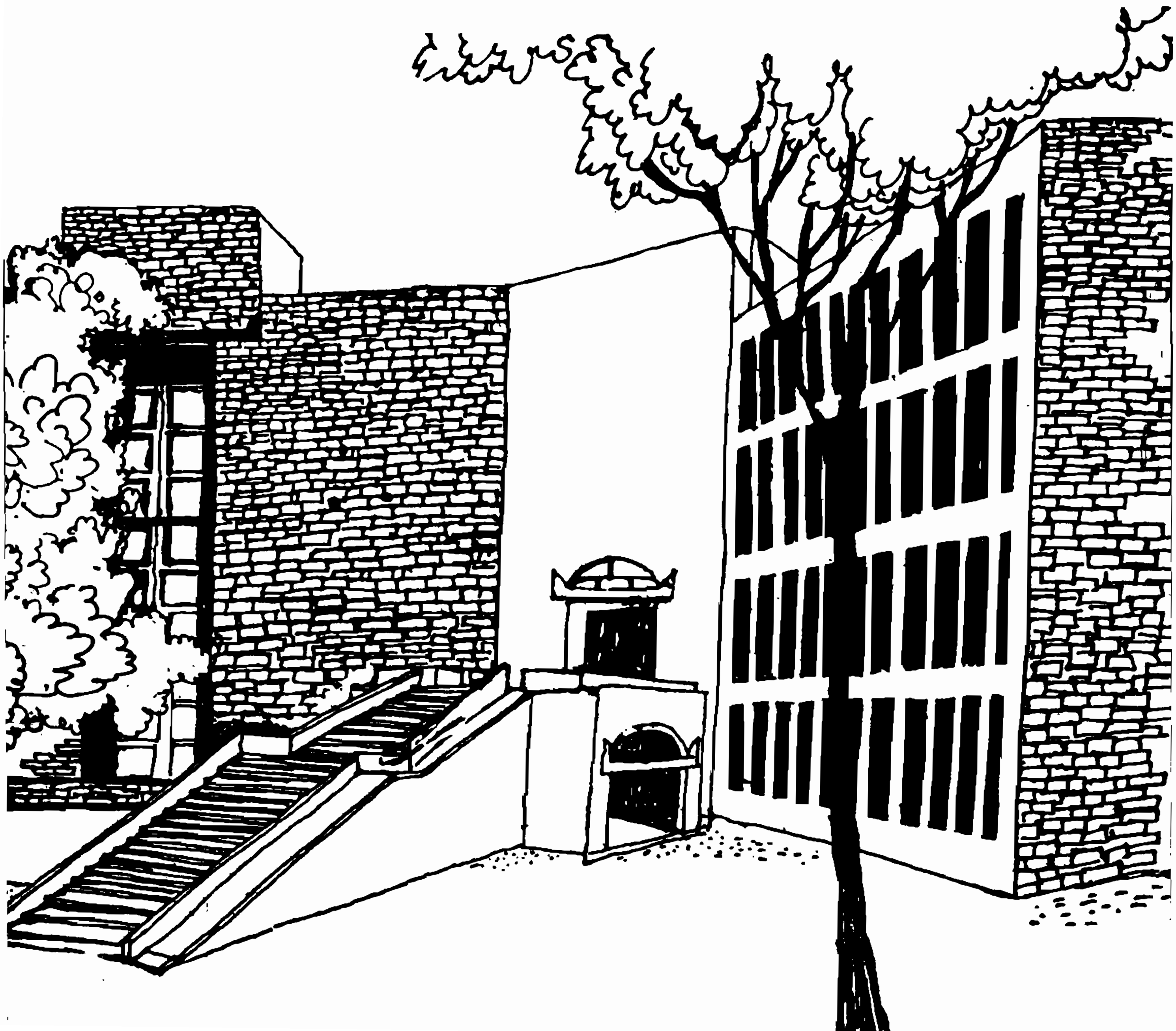

THE LEXICOGRAPHIC COMPOSITION OF ABSTRACT GAMES

By

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ABSTRACT

Two stage selection procedures are quite common. Decisions arrived at on the basis of the composition of the binary relations in some form, is what is implemented in such two stage choice procedures. The resulting binary relation is referred to as a lexicographic composition of the individual binary relations.

In this paper we begin by obtaining a necessary and sufficient condition for a quasi transitive binary relation to be transitive. Then we obtain necessary and sufficient conditions for the lexicographic composition of two quasi transitive binary relations to be quasi transitive. In passing it is noted that the lexicographic composition of two transitive binary relations is always transitive. Finally, we obtain conditions for the lexicographic composition of two binary relations to be acyclic. It is observed that if the second stage binary relation is acyclic, then the lexicographic composition is acyclic if and only if the first stage binary relation is. All our binary relations are assumed to be reflexive and complete. Such binary relations are called abstract games.

The Lexicographic Composition of Abstract Games

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1. Introduction :- Two stage selection procedures are quite common. For instance to be able to graduate in a two year program, it is often the requirement that one successfully completes the first year and then follows it by a successful completion of the second year. In interviews we often find and initial short list of candidates whose applications are reviewed in a second round of screening. The criteria by which candidates are selected at each stage is summarized in a binary relation. Decisions arrived at on the basis of the composition of the binary relations in some form, is what is implemented in such two stage choice procedures. Aleskerov (1999) refers to the resulting binary relation as a lexicographic composition of the individual binary relations.

A study of optimization based on the lexicographic composition of quasi transitive binary relations (i.e. binary relations which are representable by a vector valued function, such that one alternative dominates another alternative according to the binary relation if and only if the vector assigned to the first alternative is component wise greater than the vector assigned to the second alternative;(see Lahiri (2000 a) for a discussion of this and similar results)) was initiated by Aizerman and Malishevsky (1986) and followed up in Lahiri (2000 b).

In this paper we begin by obtaining a necessary and sufficient condition for a quasi transitive binary relation to be transitive. Then we obtain necessary and sufficient conditions for the lexicographic composition of two quasi transitive binary relations to be quasi transitive. In passing it is noted that the lexicographic composition of two transitive binary relations is always transitive. Finally, we obtain conditions for the lexicographic composition of two binary relations to be acyclic. It is observed that if the second stage binary relation is acyclic, then the lexicographic composition is acyclic if and only if the first stage binary relation is. All our binary relations are assumed to be reflexive and complete. Such binary relations are called abstract games. A comprehensive introduction to binary relations and state of the art results in the same and related areas can be found in both Aizerman and Aleskerov (1995) and Aleskerov (1999).

2. The Framework :- Let X be a non-empty finite set and let [X] denote the set of all non-empty subsets of X. Given a binary relation R on X, let $P(R) = \{(x,y) \in R:(y,x)\notin R\}$ and let $I(R) = \{(x,y) \in R:(y,x)\in R\}$. P(R) is called the asymmetric part of R and I(R) is called the symmetric part of R. Let $\Delta(X) = \{(x,x):x\in X\}$. $\Delta(X)$ is called the diagonal of X. A binary relation R on X is said to be :

(i) reflexive if $\Delta(X)\subset R$;

(ii) complete if $(X\times X) \setminus \Delta(X) \subset P(R)\cup I(R)$;

(iii) an abstract game if it is both reflexive and transitive;
(iv) acyclic if for all positive integers n and x(1),....,x(n) in X :
$[\forall i \in \{1,...,n-1\}:(x(i),x(i+1)) \in P(R)]$ implies $[(x(n),x(1)) \notin P(R)]$;
(v) transitive if $\forall x,y,z \in X$: $[\{(x,y),(y,z)\} \subset R]$ implies $[(x,y) \in R]$;
(vi) quasi transitive if P(R) is transitive.
Given a binary relation R (on X) and a set $H \in [X]$, let M(H,R) denote the set $\{x \in H$: there does not exist $y \in H$ with $(y,x) \in P(R)\}$.
The following well known proposition is available in Kreps (1988):
Proposition 1: Let R be a binary relation on X. Then $[\forall H \in [X] : M(H,R) \neq \phi]$ if and only if R is acyclic.

Let B(X) denote the class of abstract games on X,A(X) the class of acyclic abstract games on X, Tr.(X) the class of transitive abstract games and QT(X) the class of quasi transitive abstract games. Clearly, $Tr.(X) \subset QT(X) \subset A(X) \subset B(X)$.
Let $R,S \in B(X)$. The lexicographic composition of R followed by S denoted $R*S = P(R) \cup (I(R) \cap S)$. Clearly, $P(R*S) = P(R) \cup (I(R) \cap P(S))$ and $I(R*S) = (I(R) \cap I(S))$.

3. Preliminary Results:-

Theorem 1: Let $R \in QT(X)$. Then the following are equivalent :
(i) $[(x,y) \in P(R), (y,z) \in I(R)]$ implies $[(x,z) \in P(R)]$;
(ii) $[(x,y) \in I(R), (y,z) \in P(R)]$ implies $[(x,z) \in P(R)]$;
(iii) there does not exists $x,y,z, \in X$ such that $(x,y) \in I(R),(y,z) \in I(R)$ and $(x,z) \in P(R)$;
(iv) $R \in Tr.(X)$.

Proof :- (i)→(ii) : Towards a contradiction suppose $(x,y) \in I(R),(y,z) \in P(R)$ and $(x,z) \notin P(R)$. If $(z,x) \in P(R)$, then since $R \in QT(X)$ along with $(y,z) \in P(R)$, we get $(y,x) \in P(R)$ contradicting $(x,y) \in I(R)$. If $(z,x) \in I(R)$, then $(y,z) \in P(R)$ and (i) implies $(y,x) \in P(R)$, contradicting $(x,y) \in I(R)$.Hence $(x,z) \in P(R)$. Hence (i)→(ii).
(ii)→(iii) : Towards a contradiction suppose there exists $x,y,z \in X$ such that $(x,y) \in I(R)$. $(y,z) \in I(R)$ and $(x,z) \in P(R)$. Now $(y,x) \in I(R)$, $(x,z) \in P(R)$ and (ii)gives $(y,z) \in P(R)$ contradicting $(y,z) \in I(R)$. Hence (ii)→(iii).
(iii)→(iv) : Let $(x,y) \in R$ and $(y,z) \in R$. Towards a contradiction suppose $(x,z) \notin R$. Since $QT(X) \subset B(X)$, we must have $(z,x) \in P(R)$. If $(x,y) \in P(R)$, then since $R \in QT(X)$, we get $(z,y) \in P(R)$ contradicting $(y,z) \in R$. Thus, $(x,y) \in I(R)$. If $(y,z) \in P(R)$, then since $R \in QT(X)$, along with $(z,x) \in P(R)$ we get $(y,x) \in P(R)$ contradicting $(x,y) \in R$. Thus $(y,z) \in I(R)$. But then $(x,y) \in I(R)$, $(y,z) \in I(R)$ and $(z,x) \in P(R)$ contradicts (iii) (:with the roles of x and z interchanged).Thus $(x,z) \in R$. Thus (iii)→(iv).
(iv)→(i) : Suppose $(x,y) \in P(R)$, $(y,z) \in I(R)$ and towards a contradiction suppose $(x,z) \notin P(R)$. Then since $R \in QT(X) \subset B(X)$ we get $(z,x) \in R$. If $(z,x) \in P(R)$, then $(x,y) \in P(R)$ and $R \in QT(X)$ implies $(z,y) \in P(R)$, contradicting $(y,z) \in I(R)$.Hence $(z,x) \in I(R)$. But $(y,z) \in I(R)$, $(z,x) \in I(R)$ and $R \in Tr.(X)$ implies $(y,x) \in R$ contradicting $(x,y) \in P(R)$. Hence (iv)→(i).
Q.E.D.

Let $R \in B(X)$. Then an ordered triplet (x,y,z) is said to be an R-triad if $(x,y),(y,z) \in I(R)$ and $(x,z) \in P(R)$. Hence statement (iii) in Theorem 1, says that there is no R-triad. In theorem 1, we prove that a quasi-transitive abstract game R is transitive if and only if there is no R-triad.

Theorem 2 :- Let $R, S \in Tr.(X)$. Then $R*S \in Tr.(X)$.
Proof :- First let us show that $R*S \in QT(X)$.
Let $(x,y),(y,z) \in P(R*S)$. Thus $(x,y),(y,z) \in P(R) \cup (I(R) \cap P(S))$. If $(x,y),(y,z) \in P(R)$, then $(x,z) \in P(R)$, since $R \in Tr.(X)$. Thus $(x,y),(y,z) \in P(R)$ implies $(x,z) \in P(R*S)$. If $(x,y),(y,z) \in (I(R) \cap P(S))$, then $(x,z) \in I(R) \cap P(S)$, since $R,S \in Tr.(X)$.
$\therefore (x,y),(y,z) \in I(R) \cap P(S)$ implies $(x,z) \in P(R*S)$.
Suppose $(x,y) \in P(R)$ and $(y,z) \in I(R) \cap P(S)$. Thus $(x,z) \in P(R)$ since $(x,y) \in P(R)$, $(y,z) \in I(R)$ and $R \in Tr.(X)$. Hence $(x,y) \in P(R)$ and $(y,z) \in I(R) \cap P(S)$ implies $(x,z) \in P(R*S)$. Now suppose $(x,y) \in I(R) \cap P(S)$ and $(y,z) \in P(R)$. Then again $R \in Tr.(X)$ implies $(x,y) \in P(R*S)$. Thus $(x,y) \in I(R) \cap P(S)$ and $(y,z) \in P(R)$ implies $(x,y) \in P(R*S)$. Hence $R*S \in Tr.(X)$.
Now let us show that there is no $(R*S)$-triad in X . Let (x,y), $(y,z) \in I(R*S)$. Then $(x,y),(y,z) \in I(R) \cap I(S)$. Since $R,S \in Tr.(X)$ we must have $(x,z) \in I(R) \cap I(S)$. Thus $(x,z) \in I(R*S)$. Thus there is no $(R*S)$-triad in X. By Theorem 1, $R*S \in Tr.(X)$.
Q.E.D.

4. Quasitransitive Lexicographic Compositions :-

Theorem 3:- Let $R,S \in QT(X)$. Then $R*S \in QT(X)$ if and only if there does not exist an R-triad (x,y,z) such that $\{(z,y),(y,x)\} \subset S$ and $\{(z,y),(y,x)\} \cap P(S) \neq \phi$.

Proof :- Towards a contradiction suppose $R*S \in QT(X)$ and there exists an R-triad (x,y,z) with $\{(z,y),(y,x)\} \subset S$ and $\{(z,y),(y,x)\} \cap P(S) \neq \phi$. Suppose first that $(z,y) \in P(S)$. Then (a)[$(x,z) \in P(R)$ implies $(x,z) \in P(R*S)$], (b)[$(z,y) \in I(R)$ and $(z,y) \in P(S)$ implies $(z,y) \in P(R*S)$], and thus, (c) [$R*S \in QT(X)$ implies $(x,y) \in P(R*S)$]. But $(x,y) \in I(R)$ and $(x,y) \in P(R*S)$ implies $(x,y) \in P(S)$ contradicting $(y,x) \in S$. Hence there does not exist an R-triad (x,y,z) with $\{(z,y),(y,x)\} \subset S$ and $(z,y) \in P(S)$.
Now suppose (x,y,z) is an R-triad with $\{(z,y),(y,x)\} \subset S$ and $(y,x) \in P(S)$. Now (a)[$(y,x) \in I(R) \cap P(S)$ implies $(y,x) \in P(R*S)$],(b)[$(x,z) \in P(R)$ implies $(x,z) \in P(R*S)$]. Since $R*S \in QT(X)$, we get $(y,z) \in P(R*S)$. But $(z,y) \in I(R) \cap S$ implies $(z,y) \in R*S$ which contradicts $(y,z) \in P(R*S)$. Hence such an R-triad cannot exist.
Now suppose there is no R-triad such as above. Let $(x,y) \in P(R*S)$ and $(y,z) \in P(R*S)$. Towards a contradiction suppose $(x,z) \notin P(R*S)$. Hence either $(x,y) \notin P(R)$ or $(y,z) \notin P(R)$. Suppose $\{(x,y),(y,z)\} \subset I(R) \cap P(S)$. If $(x,z) \in P(R)$, then $(x,z) \in P(R*S)$. Hence $(x,z) \notin P(R)$. If $(z,x) \in P(R)$, then (z,y,x) is an R-triad with $(x,y) \in P(S)$, $(y,z) \in S$ contradicting our assumption that such is not possible . Hence $(x,z) \in I(R)$. But then $S \in QT(X)$ and $\{(x,y),(y,z)\} \subset P(S)$ implies $(x,z) \in P(S)$ and hence $(x,z) \in P(R*S)$ which is a contradiction. Hence either $(x,y) \notin I(R) \cap P(S)$ or $(y,z) \notin I(R) \cap P(S)$.

Case 1- $(x,y)\in P(R)$ and $(y,z)\in I(R)\cap P(S)$. If $(z,x)\in P(R)$, then $R\in QT(X)$ implies $(z,y)\in P(R)$ and a contradiction. If $(x,z)\in P(R)\cup(I(R)\cap P(S))$, then $(x,z)\in P(R*S)$ which leads to a contradiction. Thus, $(x,z)\in I(R)\setminus P(S)$. Hence (x,z,y) is an R-triad with $(y,z)\in P(S)$ and $(z,x)\in S$. This contradicts the assumption about the non-existence of such triples.
Case 2:- $(x,y)\in I(R)\cap P(S)$ and $(y,z)\in P(R)$. If $(x,z)\in P(R)\cup(I(R)\cap P(S))$, then $(x,z)\in P(R*S)$ leading to a contradiction. Thus $(x,z)\in I(R)\setminus P(S)$. Now (y,x,z) is an R-triad with $(z,x)\in S$ and $(x,y)\in P(S)$, which is ruled out by hypothesis.
Thus $R*S\in QT(X)$.
Q.E.D.

Corollary 1 to Theorem 3:- Let $R\in Tr.(X)$ and $S\in QT(X)$. Then $R*S\in QT(X)$.
Proof :- Follows directly from the non-existence of an R-triad if $R\in Tr.(X)$.
Q.E.D.
Given $R,S\in B(X)$ an ordered triple (x,y,z) is said to be an R-S complex if (x,y,z) is an R-triad such that $\{(z,y),(y,x)\}\subset S$ and $\{(z,y),(y,x)\}\cap P(S)\neq\phi$. Hence Theorem 3 above says that given $R,S\in QT(X)$, $R*S\in QT(X)$ if and only if there does not exist an R-S complex.

5. Acyclic Lexicographic Compositions :-

Theorem 4:- Let $R\in B(X)$ and $S\in A(X)$. Then $[R*S\in A(X)]$ if and only if $[R\in A(X)]$.
Proof :- Given $H\in[X]$, $M(H,R*S)=\{x\in H/\text{there does not exist } y\in H \text{ with } (y,x)\in P(R*S)\}$ $=\{x\in H/\text{there does not exist } y\in H \text{ with } (y,x)\in P(R)\cup(I(R)\cap P(S))\} = M(M(H,R),S)$.
Now, by Proposition 1, $[M(H,R*S)\neq\phi$, whenever $H\in[X]]$ if and only if $[R*S\in A(X)]$.
Further, $[M(H,R*S)\neq\phi$, whenever $H\in[X]]$ if and only if $[M(M(H,R),S)\neq\phi$, whenever $H\in[X]]$. But, $[M(M(H,R),S)\neq\phi$, whenever $H\in[X]]$ if and only if $[M(H,R)\neq\phi$, whenever $H\in[X]]$. This is because, by Proposition 1, $[S\in A(X)]$ implies $[M(H,S)\neq\phi$ whenever $H\in[X]]$.
Hence, by Proposition 1, $[M(H,R)\neq\phi$ whenever $H\in[X]]$ if and only if $[R\in A(X)]$. This proves the theorem.
Q.E.D.

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