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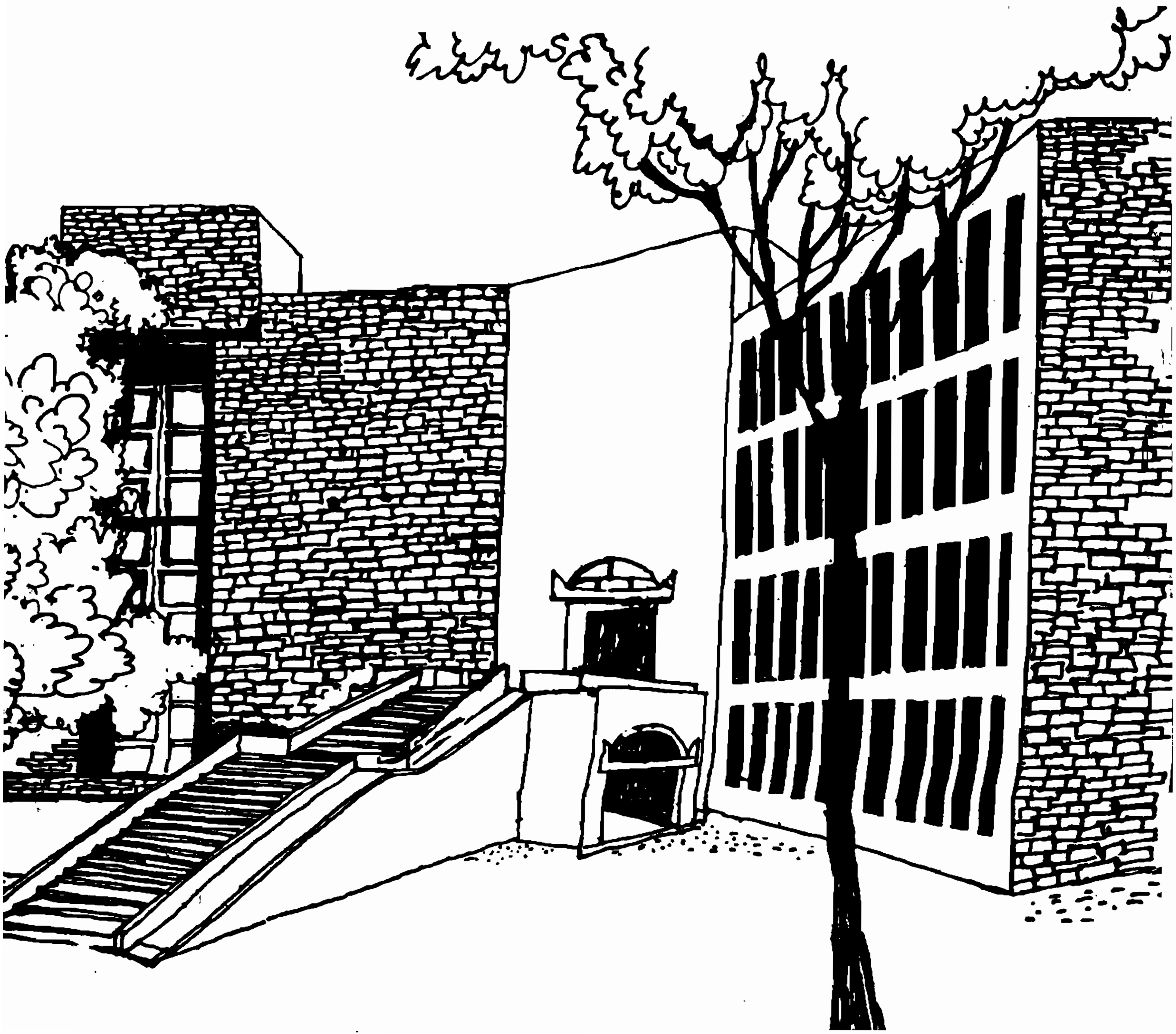

AXIOMATIC CHARACTERISATION OF
WEIGHTED VOTING OPERATOR

By

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ABSTRACT

A federation voting operator allows a finite set of coalitions to unilaterally elect any outcome. There are several special types of federation voting operators, all of which share a property :the candidates are assigned weights, and for a coalition to be decisive,it is necessary that the sum of the weights of its members exceed a pre-assigned quota. In this paper we address the following question: When is a Federation Voting Operator a Weighted Voting Operator ?

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1 Introduction

A model for analysing voting procedures where each individual in a society casts a ballot and a voting operator aggregates the ballots into elected outcomes has been modelled in Lahiri (1999,2000 a).A ballot is a set of alternatives chosen from a universal set of candidates.A ballot profile associates with each voter a ballot.A voting operator,selects a set of candidates from amongst those who have secured at least one vote.Further we assume that if there is atleast one candidate who secures the vote of every individual,then atleast one such candidate is definitely chosen.In Aczel and Roberts (1989), one is introduced to the idea of a merging function which aggregates ballots which are singletons into a singleton outcome.This is definitely a more realistic model of democratic exercises as we see it in practise. However,even though singleton ballots are a realistic premise for analysis,it is difficult to be theoretically sound and yet exclude the possibility of more than one elected outcome.Thus for instance,under plurality it is quite possible that two candidates receive the maximum number of votes.To accommodate such possibilities, Lahiri (2000 b) introduces the concept of a vote aggregator. A vote aggregator is required to satisfy the rather innocuous assumption called unanimity;i.e. if every one votes for the same candidate then that candidate is elected.It is worth recalling in this context the seminal work of Arrow,where individuals are required to vote not for a single candidate,but for a preference ordering over the entire array of candidates.This and the related literature find a thorough discussion in Aleskerov (1999).Essentially what each voter votes for is a binary relation.These binary relations are aggregated into a single binary relation.Since a binary relation is nothing but a subset of the set of all ordered pairs of candidates, the classical framework of Arrow is more appropriately a special case of the scenario where ballots are sets instead of singletons.This being the motivation behind the present work,we concentrate here on voting operators.

The voting operator we study in this paper, namely the federation voting operator originates in the work of Aizerman and Aleskerov (1986,1995).Aleskerov (1999), contains an exhaustive discussion of the related literature. A federation voting operator allows a finite set of coalitions to unilaterally elect any outcome.Such coalitions are called minimal decisive coalitions.There are several special types of federation voting operators, all of which share a property :the candidates are assigned weights, and for a coalition to be decisive,it is necessary that the sum of the weights of its members exceed a pre-assigned quota. First, there are those federation voting operators where coalitions can unilaterally elect outcomes if and only if they have a requisite number of voters. A real world example of such a voting operator is the electoral process used in electing members of the *Rajya Sabha* i.e. the upper house of the Indian Parliament. An electoral college comprising of parliamentarians who are themselves elected on the basis of universal adult franchise, must cast a certain minimum number of votes in favour of a candidate for the latter to gain entry into the *Rajya Sabha.* A second type of federation voting operator is an oligarchy, where the ability to unilaterally elect an outcome is invested in a single coalition. Finally,there is the type of federation voting operator where the ability to unilaterally elect an outcome is invested in a single individual. Such voting operators are called dictatorial voting operators.

In this paper we address the following question: When is a Federation Voting Operator a Weighted Voting Operator ?In the process of answering this question we exploit the formal similarity of a federation voting operator with a simple game due to Shapley (1962) and the formal similarity of a weighted voting operator with a weighted voting game.The unique property which is necessary and sufficient for a federation voting operator to be a weighted voting operator is called robustness in this paper.This property is similar to the concept of trade robustness that was introduced by Taylor and Zwicker (1992), and which shown by them to be necessary and sufficient for a simple game to be a weighted voting game.Our proof is completely different from the one in Taylor and Zwicker (1992).

The analytical framework in which aggregation rules are studied in this paper is similar to a device which is referred to in classical choice theory as a choice function. A comprehensive survey of rational choice theory (i.e. the theory concerned with specifying conditions on a choice function under which there exists a binary relation of a desired type whose "best" elements from a given set of alternatives, coincide with the elements chosen by the choice function) till the mid nineteen eighties is available in Moulin (1985).

2 The Model

Let n be a natural number. Let $N = \{1,...,n\}$ be the set of agents or voters. Let X be a non-empty, finite universal set of alternatives.Let P(X) denote the power set of X, i.e. the set of all subsets of X.
Let $P(X)^N$ denote the set of all functions from N to P(X).Any element $S =(S_1,...,S_n) \in P(X)^N$, is called a <u>ballot profile</u>.
A voting operator is a function $C : P(X)^N \rightarrow P(X)$ such that for all $S \in P(X)^N$: (1) $C(S) \subset \bigcup_{j\in N} S_i$; (2) $[\bigcap_{i\in N} S_i \neq \phi]$ implies $[C(S) \cap (\bigcap_{i\in N} S_i) \neq \phi]$.
Thus an element which appears on no ballot is never chosen and atleast one element which appears on the ballot of every individual is invariably chosen. As a consequence of the definition of a voting operator it easily follows that given any $x \in X$, there exists $S \in P(X)^N$ such that $\{x\} = C(S)$: simply take $\forall\, i \in N, S_i =\{x\}$.
For, $x \in S \in P(X)^N$, let $W(x,S) = \{i \in N / x \in S_i\}$.
In the sequel we will be considering the following properties of voting operators :

Monotonicity : Let $x \in C(S)$ and let S and $T \in P(X)^N$ with $\{i \in N / x \in S_i\} \subset \{i \in N / x \in T_i\}$. Then $x \in C(T)$.

Neutrality with regard to options : Let $x, y \in X$ and $S,T \in P(X)^N$. Suppose $\{i \in N/x \in S_i\} = \{i \in N/y \in T_i\}$.Then $x \in C(S) \leftrightarrow y \in C(T)$.

Robustness : Let,S and $T \in P(X)^N$, and suppose $\cup \{W(x,S)/x \in C(S)\} = \cup \{W(x,T)/x \in C(S)\}$.Suppose that $\forall i \in N: \#\{x \in C(S) / i \in W(x,S)\} = \#\{x \in C(S) / i \in W(x,T)\}$. Then, $C(S) \cap C(T) \neq \phi$.

Given a collection $\Omega = \{w_1,...,w_q\}$ of nonempty subsets of N, let $P(\Omega) = \{ w' \subset N/ w \subset w'$, for some $w \in \Omega\}$ and $L(\Omega) = \{w' \subset N / w' \notin P(\Omega)\}$. Clearly, $L(\Omega)$ contains the empty set.

<u>Definitions of Voting Operators</u> :

a) C is said to be a federation voting operator if there exists $\Omega = \{w_1,...,w_q\}$, a collection of nonempty subsets of N, such that $\forall S \in P(X)^N : C(S) = \{x \in X / W(x,S) \in P(\Omega)\}$.

b) C is said to be an oligarchy if C is a federation voting operator with $\Omega = \{ w_1 \}$.

c) C is said to be a k-votes operator (: where 'k' is a positive integer with $k \leq n$) if C is a federation voting operator with $\Omega = \{ w \subset N / w$ has exactly k elements$\}$. (A k-votes operator selects only those elements which appear on at least k-ballots.)

d) C is said to be dictatorial if there exists $i \in N$ (: called a dictator) such that $\forall S \in P(X)^N : C(S) = S_i$.

e) C is said to be a weighted voting operator if it is a federation voting operator for which there exists a function $v : N \rightarrow \aleph \cup \{0\}$ (:where $\aleph$ is the set of natural numbers) and a natural number κ (called the quota) such that $[w \in P(\Omega)]$ if and only if $[w \neq \phi, w \subset N$ and $\sum_{i \in w} v(i) \geq \kappa]$.

The following theorem has been proved in Lahiri (2000 a):

Theorem 1 : A voting operator satisfies monotonicity and neutrality with regard to options if and only if it is a federation operator.

The following observation is easy to verify:

Proposition 1 : Let C be a weighted voting operator.Then C satisfies robustness.

Proof : Let C be a federation voting operator such that $[w \in P(\Omega)]$ if and only if $[w \neq \phi, w \subset N$ and $\sum_{i \in w} v(i) \geq \kappa]$, where $v : N \rightarrow \aleph \cup \{0\}$ and κ is a natural number. Let,S and $T \in P(X)^N$, and suppose $\cup \{W(x,S)/x \in C(S)\} = \cup \{W(x,T)/x \in C(S)\}$. Suppose that $\forall i \in N: \#\{x \in C(S) / i \in W(x,S)\} = \#\{x \in C(S) / i \in W(x,T)\}$. Thus $[\sum_{i \in W(x,S)} v(i) \geq \kappa]$ for all $x \in C(S)$.Now,

$[\#\{x \in C(S) / i \in W(x,S)\}\kappa \leq \sum_{x \in C(S)} \sum_{i \in W(x,S)} v(i) = \sum_{i \in N} \#\{x \in C(S) / i \in W(x,S)\} v(i)]$. Thus,

$[\#\{x \in C(S) / i \in W(x,S)\}\kappa \leq \sum_{i \in W(x,T)} \sum_{i \in N} \#\{x \in C(S) / i \in W(x,T)\} v(i)]$,since $\forall i \in N: \#\{x \in C(S) / i \in W(x,S)\} = \#\{x \in C(S) / i \in W(x,T)\}$.Thus,

$[\#\{x \in C(S) / i \in W(x,T)\}\kappa \le \sum_{x\in C(S)} \sum_{i\in W(x,T)} v(i)]$. Hence $[\sum_{i\in W(x,T)} v(i) \ge \kappa]$ for some $x \in C(S)$.

Thus, $W(x,T) \in P(\Omega)$ for some $\in P(\Omega)$. Thus, $C(S) \cap C(T) \neq \phi$. Q.E.D.

The Characterisation Theorem

In what follows we will make the following assumption:

Assumption : $\# X \ge 2^n$.

The following proposition is immediate:

Proposition 2: Let C be a federation voting operator satisfying robustness. Suppose that $\forall\, S \in P(X)^N : C(S) = \bigcup_{j=1}^{q} \bigcap_{i\in w_j} S_i$ .Let $W_1,\ldots,W_k$ be disjoint sets in $P(\Omega)$ and let $V_1,\ldots,V_k$ be disjoint sets in N such that $\bigcup_{j=1}^{k} W_i = \bigcup_{j=1}^{k} V_i$ and $N \setminus \bigcup_{j=1}^{k} W_i$ does not belong to $P(\Omega)$.Then there exists $i \in \{1,\ldots,k\}$ such that V_i belongs to $P(\Omega)$.

Proof : Let $X = \{x_1,\ldots, x_m\}$, where $m = 2^n$, and let $S \in P(X)^N$,with $W(x_i,S) = W_i$ for $i \in \{1,\ldots,k\}$ and $W(x_m,S) = N \setminus \bigcup_{j=1}^{k} W_i$. Thus $C(S) = \{x_1,\ldots, x_k\}$. Let $T \in P(X)^N$, with $W(x_i,T) = V_i$ for $i \in \{1,\ldots,k\}$ and $W(x_m,T) = N \setminus \bigcup_{j=1}^{k} V_i = N \setminus \bigcup_{j=1}^{k} W_i$. By robustness, $C(S) \cap C(T) \neq \phi$. Thus there exists $i \in \{1,\ldots,k\}$ such that V_i belongs to $P(\Omega)$.

Q.E.D.

The following lemma is crucial in what follows:

<u>Lemma 1</u> :- Let $\sum_{j=1}^{n} a_{ij}x_j = b_i$, i=1,...,k be a system of 'k' equation in 'n' unknowns and suppose a_{ij}, b_i are all rational for i=1,..,k ; j=1,...,n. Let $(x_1^*,\ldots,x_n^*)$ be a solution for the above system of equations. Then given ε>0, there exists a solution $(\bar{x}_1,\ldots,\bar{x}_n)$ with all co-ordinates rational such that $\|(x_1^*,\ldots,x_n^*)(\bar{x}_1,\ldots,\bar{x}_n)\| < \varepsilon$.

<u>Proof</u> :- If n=1, then $a_{i1}x_1^* = b_i$, i=1,...,k with all a_{i1}, b_i, i=1,...,k rational implies $x_1^* = \frac{b_i}{a_{i1}}$ whenever $a_{i1} \neq 0$. If $a_{i1}=0\,\forall i$, then $b_i=0$ $\forall i$ and hence we can choose any $\bar{x}_1 \in (x_1^* - \varepsilon, x_1^* + \varepsilon)\bar{x}_1$ rational to solve the system. In either case the theorem is true for n=1. Suppose the theorem is true for 1,2,...,n-1 where n-1≥0. Let $\sum_{j=1}^{n} a_{ij}x_j = b_i$, i=1,...,k be the system as desired and let $(x_1^*,\ldots,x_n^*)$ solve the system. Without loss of generality suppose $a_{kn} \neq 0$. Let $x_n = \frac{1}{a}\left[b_k - \sum_{j=1}^{n-1} a_{kj}x_j\right]$. Since the real valued

function $(y_1,\ldots,y_{n-1}) \mapsto \frac{1}{a_k}\left[b_k - \sum_{j=1}^{n-1} a_{kj}y_j\right]$,with domain $\Re^{n-1}$ is continuous,there exists $\delta>0$: $\left\|(y_1,\ldots,y_{n-1})-(x_1^*,\ldots,x_{n-1}^*)\right\| < \delta \rightarrow \left|\frac{1}{a_k}\left[b_k - \sum a_{kj}y_j\right] - x_n^*\right| < \frac{\varepsilon}{n}$.

Consider the system,

$\sum_{j=1}^{n-1} C_{ij}x_j = B_i$, i=1,...,k where $C_{kj} = 0 = B_k$, for j=1,..,n-1 and $C_{ij}=a_{ij}-\frac{a_{in}}{a_{kn}}a_{kj}$,

$B_i=b_i - \frac{b_k}{a_{kn}}$ for i=1,...,k-1 , j=1,..,n-1.

Now $\sum_{j=1}^{n-1} a_{ij}x_j^* + \frac{a_{in}}{a_{kn}}\left[b_k - \sum_{j=1}^{n-1} a_{kj}x_j^*\right] = b_i$ for i=1,...,k-1

$\therefore \sum\left(a_{ij} - \frac{a_{in}}{a_{kn}}a_{kj}\right)x_j^* = b_i - \frac{a_{in}}{a_{kn}}b_k$ for i=1,...,k-1.

$\therefore (x_1^*,\ldots,x_{n-1}^*)$ satisfies the new system. By the induction hypothesis there exists $(\bar{x}_1,\ldots,\bar{x}_{n-1})$ with all co-ordinates rational such that

$\left\|(\bar{x}_1,\ldots,\bar{x}_{n-1})-(x_1^*,\ldots,x_{n-1}^*)\right\| < \min\left\{\frac{\varepsilon}{2},\delta\right\}$. Let $\bar{x}_n = \frac{1}{a_k}\left[b_k - \sum_{j=1}^{n-1} a_{kj}\bar{x}_j\right]$. Clearly $\bar{x}_n$ is rational since it is obtained from $\|(\bar{x}_1,\ldots,\bar{x}_{n-1})$. Further, $\left|\bar{x}_n - x_n^*\right| < \frac{\varepsilon}{n}$.

$\therefore \left\|(\bar{x}_1,\ldots,\bar{x}_n)-(x_1^*,\ldots,x_n^*)\right\|^2 = \sum_{j=1}^{n}(\bar{x}_j - x_j^*)^2 = \sum_{j=1}^{n-1}(\bar{x}_n - x_n^*)^2 < \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{n^2} \le \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{2}$.

$\therefore \left\|(\bar{x}_1,\ldots,\bar{x}_n)-(x_1^*,\ldots,x_n^*)\right\|^2 < \frac{\varepsilon}{\sqrt{2}} < \varepsilon$. Now $\bar{x}_n = \frac{1}{a_{kn}}\left[b_k - \sum_{j=1}^{n-1} a_{kj}x_j\right]$ implies

$\sum_{j=1}^{n} a_{kj}\bar{x}_j = b_k$. For i<k,

$\sum_{j=1}^{n-1}\left(a_{ij} - \frac{a_{in}}{a_{kn}}a_j\right)\bar{x}_j = b_i - \frac{a_{in}}{a_{kn}}b_k$. $\sum_{j=1}^{n-1} a_{ij}\bar{x}_j - \sum_{j=1}^{n-1}\frac{a_{in}}{a_{kn}}a_j\bar{x}_j = b_i - \frac{a_{in}}{a_{kn}}b_k$ or

$\sum_{j=1}^{n-1} a_{ij}\bar{x}_j + a_{in}\left[\frac{b_k}{a_{kn}} - \sum_{j=1}^{n-1} a_{kj}\bar{x}_j\right] = b_i$ i.e., $\sum_{j=1}^{n} a_{ij}\bar{x}_j = b_i$. Hence if the theorem is assumed true for 1,...,n-1 with n-1≥0, then it is true for n. We have already shown that it is true for 1. Hence it is true for all n.

Q.E.D.

Proposition 3 : Let C be a federation voting operator which satisfies robustness. Then, C is a weighted voting operator.

Proof :- We prove this proposition by induction on n = # N.If N is a singleton, we have N={1}.By unanimity, {1} = Ω.Let v:N→$\aleph \cup\{0\}$ be defined by v(1)=1 and let κ=1. Then clearly, [w∈P(Ω)] if and only if [w≠ϕ, w⊂N and $\sum_{i\in w} v(i) \geq \kappa$].

Suppose the proposition is true for #N=1,...,r-1, where r is any natural number. Let #N=r. For w⊂ N, let e_w:N→{0,1} be defined by $e_w(i)$ =1 if i∈w, $e_w(i)$ =0 , if i∉S.

Let A=$\left\{ \sum_{w\in P(\Omega)} t_w e_w \,/\, t_w \in [0,1] \forall w \in P(\Omega) \text{and} \sum_{w\in P(\Omega)} t_w = 1 \right\}$ and

B=$\left\{ \sum_{w\in L(\Omega)} t_w e_w \,/\, t_w \in [0,1] \forall w \in L(\Omega) \text{and} \sum_{w\in L(\Omega)} t_w = 1 \right\}$

Both A and B are non-empty convex subsets of $\Re^r$. Let $\chi \in$A and $\eta \in$B with $\chi = \sum_{w\in\Omega} t_w e_w = \sum_{w\in L(\Omega)} t_w e_w = \eta$ and t_w being a rational number for all w∈$\Omega \cup L(\Omega)$. By

Taking the LCM of the denominators we may assume that ∀ w∈$\Omega \cup L(\Omega)$, $t_w = \frac{n_w}{K}$ where K∈$\aleph$ and $n_w \in \aleph \cup \{0\}$. Thus $\sum_{w\in\Omega} n_w = \sum_{w\in L(\Omega)} n_w = K$. Taking n_w copies of w for each w∈$\Omega \cup L(\Omega)$ we get a violation of robustness.

Now suppose,

$$\sum_{w\in\Omega} t_w e_w - \sum_{w\in L(\Omega)} t_w e_w = 0, \text{and}$$

$$\sum_{w\in\Omega} t_w = 1 = \sum_{w\in L(\Omega)} t_w$$

has a non-negative solution.

$$\therefore \sum_{\substack{w\in\Omega \\ t_w>0}} t_w e_w - \sum_{\substack{w\in L(\Omega) \\ t_w>0}} t_w e_w = 0, \text{ and}$$

$$\sum_{\substack{w\in\Omega \\ t_w>0}} t_w = 1 = \sum_{\substack{w\in L(\Omega) \\ t_w>0}} t_w$$

has a strictly positive solution.

Hence by Lemma 1, $\sum_{\substack{w\in\Omega \\ t_w>0}} t_w e_w - \sum_{\substack{w\in L(\Omega) \\ t_w>0}} t_w e_w = 0$, and

$$\sum_{\substack{w\in\Omega \\ t_w>0}} t_w = 1 = \sum_{\substack{w\in L(\Omega) \\ t_w>0}} t_w$$

has a strictly positive solution all whose co-ordinates are rational. Thus,

$$\sum_{w\in\Omega} t_w e_w - \sum_{w\in L(\Omega)} t_w e_w = 0, \text{and}$$

$$\sum_{w\in\Omega} t_w = 1 = \sum_{w\in L(\Omega)} t_w$$

has a non-negative solution all whose co-ordinates are rational, contradicting what we obtained earlier in the proof. Thus A∩B=ϕ. By the separating hyperplane theorem, there exists p∈$\Re^N \setminus \{0\}$ such that p.χ>p.η ∀(χ,η)∈AxB. Thus p.e_w>p.$e_{w'}$ ∀(w,w')∈P(Ω)xL(Ω).

Suppose for some j∈N:p_j <0.

Case 1 :- There exists w∈ P(Ω) such that j∈w. Let w'∈w\{j}. Now p.($e_{w'}$-e_w)=p_j<0 contradicting what we obtained above.

Case 2 :- For all w∈ $P(\Omega)$, w \{j}∈ $P(\Omega)$. Without loss of generality suppose j = n. Let $\bar{\Omega} = \{w \setminus \{n\} / w \in \Omega\}$. Thus #(N\{n})=n-1 and it is easily verified that the federation voting operator $\forall$ S $\in P(X)^{N\setminus\{n\}}$: $\bar{C}$(S)= {x∈X / W(x,S)∈$P(\bar{\Omega})$} satisfies robustness. Then by the induction hypothesis , there exists a function v' :N\{n} → $\aleph\cup\{0\}$ and a natural number κ such that [w∈$P(\bar{\Omega})$] if and only if [w ≠φ, w ⊂N\{n} and $\sum_{i\in w} v'(i) \geq \kappa$]. Let v:P→$\aleph\cup\{0\}$ be defined by setting, v(i)= $v'(i) \forall i \in N \setminus \{n\}$, and v(n)=0. Then it is easily verified that [w∈P(Ω)] if and only if [w≠φ, w⊂N and $\sum_{i\in w} v(i) \geq \kappa$].

Hence p∈ $\Re_+^N \setminus \{0\}$. Clearly there exists $\bar{p} \in \Re_+^N \setminus \{o\}$ with all co-ordinates rational such that $\min\{\bar{p}.e_w / w \in \Omega\} > \max\{\bar{p}.e_w / w \in L(\Omega)\}$. By multiplying the numerators of $\bar{p}$ by the LCM of the denominators we get v :N→$\aleph\cup\{0\}$ such that $\min\{\sum_{i\in w} v(i) / w \in \Omega\} > \max\{\sum_{i\in w} v(i) / w \in L(\Omega)\}$. Let $\kappa = \min\{\sum_{i\in w} v(i) / w \in \Omega\}$. Thus [w∈P(Ω)] if and only if [w≠φ, w⊂N and $\sum_{i\in w} v(i) \geq \kappa$]. The proposition stands established by a standard induction argument.

Q.E.D.

Propositions 1 and 3 combined together, constitute a proof of the following theorem:

Theorem 2 :A federation voting operator C is a weighted voting operator if and only if C satisfies robustness.

In view of Theorems 1 and 2 the following characterisation theorem for weighted voting operators is immediate.

Theorem 3 : A voting operator C is a weighted voting operator if and only if C satisfies monotonicity, neutrality with regard to options and robustness.

Example 1: Let C(S)= {x∈X / W(x,S)∈P(Ω)},∀ S ∈$P(X)^N$, where Ω = {w} (a singleton) for some non-empty subset w of N. Clearly C is an oligarchy. Define v :N → $\aleph\cup\{0\}$ as follows:v(i)=1 if i∈w, v(i) = 0 if i∉N \ w.Let κ = #w.Then, [w∈P(Ω)] if and only if [w≠φ, w⊂N and $\sum_{i\in w} v(i) \geq \kappa$].

Example 2: Let k be a positive integer less than or equal to n and let C(S)= {x∈X / W(x,S)∈P(Ω)},∀ S ∈$P(X)^N$, where Ω = {w⊂ N / # w = k}.Clearly C is a k-votes operator. Define v :N → $\aleph\cup\{0\}$ as follows:v(i)=1 if ∀i∈N.Let κ = k.Then, [w∈P(Ω)] if and only if [w≠φ, w⊂N and $\sum_{i\in w} v(i) \geq \kappa$].

Example 3: Let i∈N and let C(S)= {x∈X / W(x,S)∈P(Ω)},∀ S ∈$P(X)^N$, where Ω = {w ⊂ N / i∈ w}.Clearly C is a dictatorial voting operator. Define v :N → $\aleph\cup\{0\}$ as follows:v(j)=1 if j=i,v(j) = 0 if j ≠ i. Let κ = 1.Then, [w∈P(Ω)] if and only if [w≠φ, w⊂N and $\sum_{i\in w} v(i) \geq \kappa$].

Note:In some senses,an oligarchy is a basic unit of any federation voting operator.For, let $C(S)=\{x\in X / W(x,S)\in P(\Omega)\}, \forall S \in P(X)^N$ where $\Omega = \{w_1,\ldots,w_q\}$ is a collection of nonempty subsets of N.For $i \in\{1,\ldots,q\}$, let $v_i : N \rightarrow \aleph\cup\{0\}$ be defined as follows: $v_i(j) = 1$ if $j \in w_i$, $v_i(j) = 0$ if $j \in N\setminus w_i$ .For $i \in\{1,\ldots,q\}$, let $\kappa_i = \#w_i$. Thus, $[w\in P(\Omega)]$ if and only if $[w\neq\phi$, $w\subset N$ and $[\exists i\in\{1,\ldots,q\}: \sum_{j\in w} v_i(j) \geq \kappa_i]\ ]$. In view of this observation,we the following theorem stands established :

Theorem 4 : Let C be a federation voting operator. Then, there exists a natural number k and weighted voting operators $C_1,\ldots,C_k$ such that $\forall S \in P(X)^N$: $C(S)= \cup \{ C_i(S) / i \in\{1,\ldots,k\}$.

Let C be a federation voting operator. Then, min $\{k / \forall S \in P(X)^N\ C(S)= \cup \{ C_i(S) / i \in\{1,\ldots,k\}\}$ is called the dimension of C, and is denoted by k(C). Clearly k(C) is always greater than or equal to one, and is equal to one if and only if C is a weighted voting operator.Thus the dimension of oligarchies, k- votes operators and dictatorial voting operators are one.However it is easy to provide examples of federation voting operators for which k(C) is greater than one.

Example 4: Let $n = 2k$ for some positive integer k. Let $\Omega = \{ w\subset N / w = \{2j-1,2j\}$, for some $j \in \{1,\ldots,k\}\}$. Let C be a federation voting operator such that $\forall S \in P(X)^N$ $:C(S)= \{x\in X / W(x,S)\in P(\Omega)\}$. Towards a contradiction suppose that C is a weighted voting operator. Then, there exists a function $v : N \rightarrow \aleph\cup\{0\}$ and a natural number κ such that $[w\in P(\Omega)]$ if and only if $[w\neq\phi$, $w\subset N$ and $\sum_{i\in w} v(i) \geq \kappa]$.

Thus, $v(1) + v(2) \geq \kappa$, $v(3) + v(4) \geq \kappa$, since both {1,2} and {3,4} belong to Ω. Hence either $v(2) + v(3) \geq \kappa$ or $v(1) + v(4) \geq \kappa$. Thus,either {2,3} or {1,4} belongs to $P(\Omega)$, contradicting our definition of Ω. Thus, C is not a weighted voting operator.

For $i \in\{1,\ldots,k\}$, let $v_i : N \rightarrow \aleph\cup\{0\}$ be defined as follows: $v_i(2i-1) = v_i(2i)=1$ $v_i(j) = 0$ if $j \in N\setminus\{2i-1,2i\}$.For $i \in\{1,\ldots,q\}$, let $\kappa_i = 2$. Thus, $[w\in P(\Omega)]$ if and only if $[w\neq\phi$, $w\subset N$ and $[\exists i\in\{1,\ldots,k\}: \sum_{j\in w} v_i(j) \geq \kappa_i]\ ]$.Infact,it is possible to establish via an induction argument that k(C) = k. For k =1,2 it is easy to verify that k(C) = k. Assume that k(C) = k, for $k =1,\ldots,r-1\geq 2$.Let, k=r and towards a contradiction suppose that k(C) < r. Thus, for $i \in\{1,\ldots,k(C)\}$ with k(C) < r, there exists functions $v_i : N \rightarrow \aleph\cup\{0\}$ and a natural number κ_i such that $[w\in P(\Omega)]$ if and only if $[w\neq\phi$, $w\subset N$ and $[\exists i\in\{1,\ldots,k(C)\}: \sum_{j\in w} v_i(j) \geq \kappa_i]\ ]$.Since, $\Omega = \{ w\subset N / w = \{2j-1,2j\}$, for some $j \in \{1,\ldots,r\}\}$, there exists $j_1, j_2 \in\{1,\ldots,r\}$ with $j_1 \neq j_2$ and $h \in\{1,\ldots,k(C)\}$, such that (a) $v_h(2j_1 - 1) + v_h(2j_1) \geq \kappa_h$; (b) $v_h(2j_2 - 1) + v_h(2j_2) \geq \kappa_h$;(c) $[\sum_{j\in w} v_h(j)\leq\kappa_h, \text{if } w \notin P(\Omega)]$. By symmetry of the problem under consideration,we may let $j_1 =1$ and $j_2 =2$.Thus for k = 2, k(C) = 1, which is not possible. Thus, k(C) = r. By a standard induction argument, it follows that k(C) = k for every natural number k.

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