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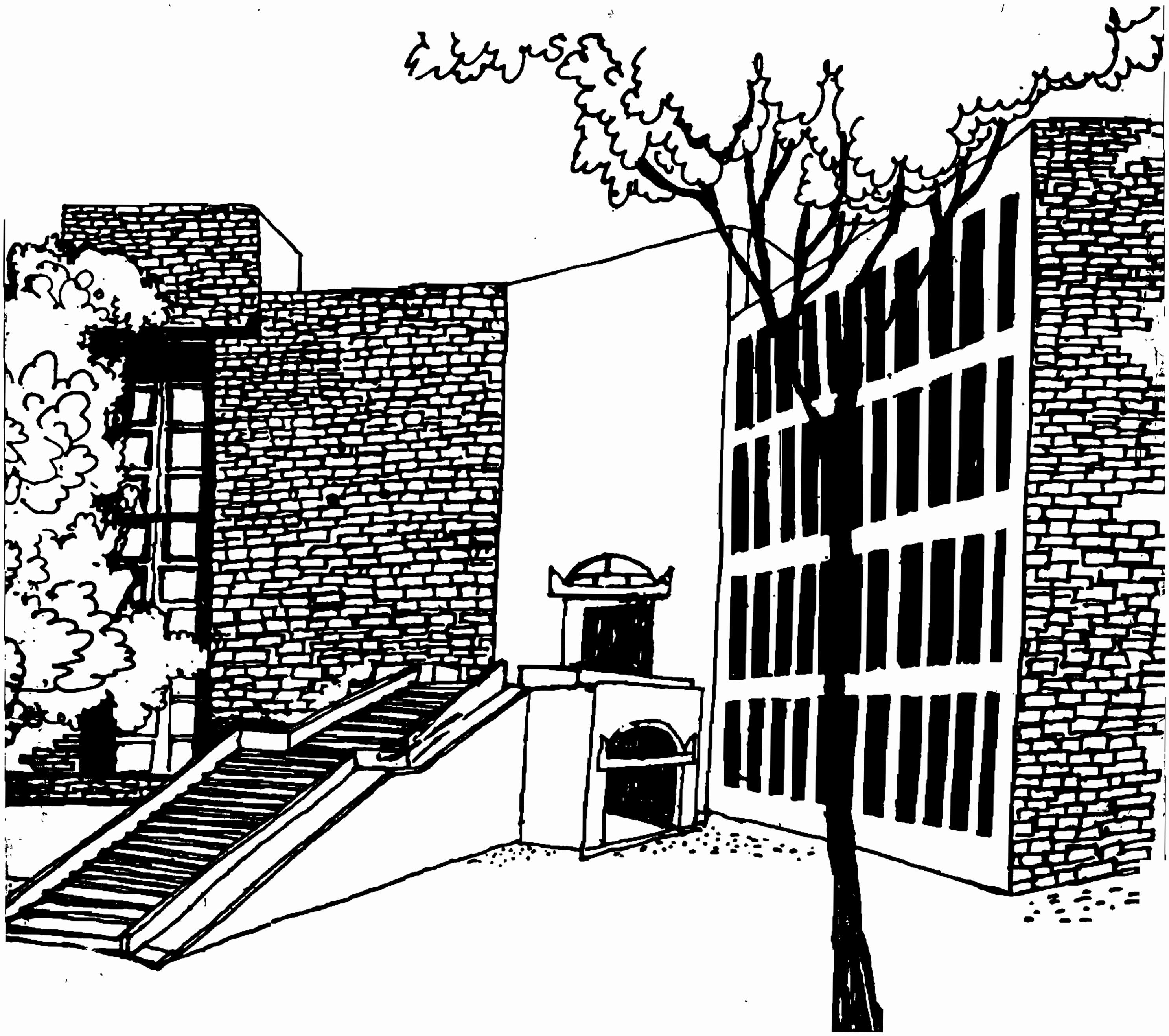

"ON DIVERSITY AND FREEDOM OF CHOICE":
A TECHNICAL COMMENT

By

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"On Diversity and Freedom of Choice": A Technical Comment

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Abstract: *In this paper we show that the similarity based preference for freedom of choice defined by Pattanaik and Xu (2000), can be uniquely characterized by Indifference Between No-Choice Situation, Strong Monotonicity with respect to the similarity relation and Weak Composition with respect to the similarity relation.*

The model and problem that we study in this paper is due to Pattanaik and Xu (2000).

Let X be a non-empty finite set of alternatives containing at least two elements. Let [X] be the set of all non empty subsets of X. Let $\Delta(X) = \{(x,x)/x\in X\}$ and $\Delta([X]) = \{(A,A)/A\in[X]\}$. $\Delta(X)$ is called the diagonal of X and $\Delta([X])$ is called the diagonal of [X].

A binary relation S on X is said to be :

(a)reflexive, if $\Delta(X) \subset S$;

(b)symmetric if $\forall x,y \in X : (x,y) \in S$ implies $(y,x) \in S$.

A binary relation S on X which is reflexive and symmetric is called a similarity relation. Let S be a fixed similarity relation.

A set $A\in[X]$ is said to be homogeneous if $A\times A \subset S$.

If a set $A\in[X]$ is not homogeneous and if $B\in[X]$ with $A \subset B$, then it is easy to see that B is not homogeneous.

Given $A\in[X]$, a similarity based partition of A is a partition f of A such that $\forall$ $B\in f$, it is true that B is homogeneous.

Example: Let $f= \{\{x\}/x\in A\}$. Then f is a similarity based partition of A and the only one, if $S = \Delta(X)$.

For $A\in[X]$,let $F(A) =\{f/$ f is a similarity based partition of A and $\#f \leq \#g$, whenever g is any other similarity based partition of A$\}$and let $n(A) = \#f$, for some $f\in F(A)$.Clearly n(A) is a well defined positive integer.

Given $A,B\in[X]$ with both A and B being homogeneous we say that A does not mimic B if $A\cup B$ is not homogeneous. Given $A,B\in[X]$ with A homogeneous we say that A does not mimic B if for all $f\in F(B)$ and all $C \in f$, it is the case that A does not mimic C.

Note that if $A\in[X]$ with $n(A) > 1$ and if $f \in F(A)$, then given any $B,C \in f$, it is true that B does not mimic C.Thus B does not mimic the union of collection of members of f different from B.

A binary relation $\Re$ on [X] is said to be :

(a) reflexive, if $\Delta([X]) \subset \Re$;

(b) complete, if given $A,B \in[X]$, with $A \neq B$, either $(A,B) \in\Re$ or $(B,A) \in\Re$;

(c) a preference for freedom of choice (PFC) if it is both reflexive and complete;

Given a binary relation $\Re$ on [X], let $P(\Re) = \{(A,B) \in\Re / (B,A) \notin\Re\}$ denote the asymmetric part of $\Re$ and let $I(\Re) = \{(A,B)\in\Re / (B,A) \in\Re\}$ denote the symmetric part of $\Re$.

A binary relation $\Re$ on [X] is said to be transitive if $\forall A,B,C \in[X]$: $[(A,B),(B,C)\in \Re]$ implies $[(A,C)\in \Re]$.

Define a binary relation $\Re^S$ on [X] as follows: $\forall A,B\in[X] : (A,B) \in\Re^S$ if and only if $n(A) \geq n(B)$.

Clearly, $\Re^S$ is a transitive PFC.

A PFC is said to satisfy:

(1) Indifference Between No-Choice Situation (INS) if $\forall x,y \in X$ it is the case that $(\{x\},\{y\}) \in I(\mathfrak{R})$;

(2) S–Monotonicity (SM) if $\forall A \in [X]$ such that A is homogeneous and for all $x \in X \setminus A$:

(a)[$A \cup \{x\}$ is homogeneous] implies [$(A \cup \{x\}, A) \in I(\mathfrak{R})$];

(b)[$A \cup \{x\}$ is not homogeneous] implies [$(A \cup \{x\}, A) \in P(\mathfrak{R})$];

(3) S-Composition (SC) if $\forall A,B,C,D \in [X]$ with $A \cap C = B \cap D = \phi$, both C and D homogeneous, C does not mimic A and $(C,D) \in \mathfrak{R}$:

(a) $(A,B) \in I(\mathfrak{R})$ implies that $(A \cup C, B \cup D) \in \mathfrak{R}$;

(b) $(A,B) \in P(\mathfrak{R})$ implies that $(A \cup C, B \cup D) \in P(\mathfrak{R})$;

(4) Strong S-Monotonicity (SSM) if $\forall A \in [X]$, $f \in F(A)$ and for all $x \in X \setminus A$:

(a)[$A \cup \{x\}$ is homogeneous] implies [$(A \cup \{x\}, A) \in I(\mathfrak{R})$];

(b)[$\forall B \in f$: $B \cup \{x\}$ is not homogeneous] implies [$(A \cup \{x\}, A) \in P(\mathfrak{R})$];

(5) Weak S-Composition (WSC) if $\forall A,B,C,D \in [X]$ with $A \cap C = B \cap D = \phi$, both C and D homogeneous, C does not mimic A and $(C,D) \in I(\mathfrak{R})$: [$(A,B) \in I(\mathfrak{R})$] implies that [$(A \cup C, B \cup D) \in \mathfrak{R}$].

Clearly SC implies WSC.

Claim 1: If a PFC $\mathfrak{R}$ satisfies SM and SC, then it satisfies SSM.

Proof: Part (a) of SSM is contained in the definition of SM. Hence we have to show that if a PFC $\mathfrak{R}$ satisfies SM and SC, then it satisfies part (b) of SSM. Suppose that $\mathfrak{R}$ is a PFC which satisfies SM and SC and let $A \in [X]$, $f \in F(A)$, $x \in X \setminus A$:[$\forall B \in f$: $B \cup \{x\}$ is not homogeneous]. Let $f = \{A(1),\ldots,A(k)\}$ with $k > 1$. Since $A(1) \cup \{x\}$ is not homogeneous, SM implies that $(A(1) \cup \{x\}, A(1)) \in P(\mathfrak{R})$. Suppose, that for some $j \in \{1,\ldots,k-1\}$ it is the case that $(A(1) \cup \ldots \cup A(j) \cup \{x\}, A(1) \cup \ldots \cup A(j)) \in P(\mathfrak{R})$. Since $A(j+1)$ is homogeneous, $(A(1) \cup \ldots \cup A(j) \cup \{x\}) \cap A(j+1) = \phi$, and since $f \in F(A)$ implies that $A(1) \cup \ldots \cup A(j) \cup A(j+1)$ (:and hence $(A(1) \cup \ldots \cup A(j) \cup A(j+1) \cup \{x\})$ is not homogeneous, by SC it follows that $(A(1) \cup \ldots \cup A(j) \cup A(j+1) \cup \{x\}, A(1) \cup \ldots \cup A(j) \cup A(j+1)) \in P(\mathfrak{R})$. By a standard induction argument it follows that $(A(1) \cup \ldots \cup A(j) \cup \{x\}, A(1) \cup \ldots \cup A(j)) \in P(\mathfrak{R})$ for all $j \in \{1,\ldots,k\}$. Thus $(A \cup \{x\}, A) \in P(\mathfrak{R})$. Q.E.D.

Theorem 1(Pattanaik and Xu (2000)): Let $\mathfrak{R}$ be a transitive PFC. $\mathfrak{R}$ satisfies INS,SM,SC if and only if $\mathfrak{R} = \mathfrak{R}^S$.

Theorem 2: Let $\mathfrak{R}$ be a transitive PFC. $\mathfrak{R}$ satisfies INS,SSM,WSC if and only if $\mathfrak{R} = \mathfrak{R}^S$.

We will prove Theorem 2 by putting together the conclusions of several propositions.

Proposition 1: Let $\mathfrak{R}$ be a transitive PFC. Suppose $\mathfrak{R}$ satisfies SM, $A \in [X]$ is homogeneous and $x \in A$. Then $(A, \{x\}) \in I(\mathfrak{R})$.

Proof: Let $\mathfrak{R}$, A and x as in the proposition. If $A = \{x\}$, then the proposition follows from the reflexivity of $\mathfrak{R}$. Hence suppose that $A = \{x(1),\ldots,x(k)\}$ for some $k > 1$, with $x(1) = x$. Clearly, $(\{x(1)\},\{x(1)\}) \in I(\mathfrak{R})$. Suppose that for some $j \in \{1,k-1\}$: $(\{x(1),\ldots,x(j)\},\{x(1)\}) \in I(\mathfrak{R})$. Since $\{x(1),\ldots,x(j),x(j+1)\}$ is homogeneous, by SM we get $(\{x(1),\ldots,x(j),x(j+1)\}, \{x(1),\ldots,x(j)\}) \in I(\mathfrak{R})$. By transitivity of $\mathfrak{R}$, we get that $(\{x(1),\ldots,x(j),x(j+1)\}, \{x(1)\}) \in I(\mathfrak{R})$. By a standard induction argument it now follows that $(A,\{x(1)\}) \in I(\mathfrak{R})$. Thus, $(A,\{x\}) \in I(\mathfrak{R})$. Q.E.D.

Proposition 2: Let $\mathfrak{R}$ be a transitive PFC. Suppose $\mathfrak{R}$ satisfies INS and SM. Let $A,B \in [X]$ and suppose A and B are homogeneous. Then $(A,B) \in I(\mathfrak{R})$.

Proof: Follows immediately from Proposition 1, INS and transitivity of $\mathfrak{R}$. Q.E.D.

Proposition 3: Let $\mathfrak{R}$ be a transitive PFC, which satisfies INS,SM and SC .Let $A,B \in [X]$ and suppose $n(A) = n(B)$. Then $(A,B) \in I(\mathfrak{R})$.

Proof: Let $\mathfrak{R}$,A and B be as in the proposition. Supose that $f = \{A(1),\ldots,A(k)\} \in F(A)$ and $g = \{B(1),\ldots,B(k)\} \in F(B)$. By Proposition 2, $(A(1),B(1)) \in I(\mathfrak{R})$.If $k = 1$, then the proposition stands established. Hence suppose that $k > 1$. Suppose that for some $j \in \{1,\ldots,k-1\}$ it is the case that $(A(1) \cup \ldots \cup A(j), B(1) \cup \ldots \cup B(j)) \in I(\mathfrak{R})$. Since $A(j+1) < B(j+1)$ are homogeneous, $(A(1) \cup \ldots \cup A(j)) \cap A(j+1) = \phi = (B(1) \cup \ldots \cup B(j)) \cap B(j+1)$,and since $f \in F(A), g \in F(B)$ implies that $A(1) \cup \ldots \cup A(j) \cup A(j+1)$ and $B(1) \cup \ldots \cup B(j) \cup B(j+1)$ are not homogeneous, a double and symmetric application of WSC implies that $(A(1) \cup \ldots \cup A(j+1), B(1) \cup \ldots \cup B(j+1)) \in I(\mathfrak{R})$. A standard induction argument now implies that $(A,B) \in I(\mathfrak{R})$. Q.E.D.

Proposition 4 : Let $\mathfrak{R}$ be a transitive PFC, which satisfies INS,SM and SC. Suppose $A \in [X]$ and that $f = \{A(1),\ldots,A(k)\} \in F(A)$ for some $k > 1$. Then, $(A, \bigcup_{i=1}^{k-1} A(i)) \in P(\mathfrak{R})$.

Proof : Let $x \in A_k$ such that whenever $i \in \{1,\ldots,k-1\}$, $A(i) \cup \{x\}$ is not homogeneous. Such an x exists since $f \in F(A)$. By Proposition 1, $(A(k),\{x\}) \in I(\mathfrak{R})$.By WSC, $(A, \bigcup_{i=1}^{k-1} A(i) \cup \{x\}) \in \mathfrak{R}$. Since $\{A(1),\ldots,A(k-1)\} \in F(\bigcup_{i=1}^{k-1} A(i))$, by SSM we get $(\bigcup_{i=1}^{k-1} A(i) \cup \{x\}, \bigcup_{i=1}^{k-1} A(i)) \in P(\mathfrak{R})$. By transitivity of $\mathfrak{R}$, we obtain $(A, \bigcup_{i=1}^{k-1} A(i)) \in P(\mathfrak{R})$.

Q.E.D.

Proposition 5: Let $\mathfrak{R}$ be a transitive PFC, which satisfies INS,SSM and WSC .Let $A,B \in [X]$ and suppose $n(A) > n(B)$. Then $(A,B) \in P(\mathfrak{R})$.

Proof: Let $\mathfrak{R}$,A and B be as in the proposition. Supose that $f = \{A(1),\ldots,A(m)\} \in F(A)$ and $g = \{B(1),\ldots,B(k)\} \in F(B)$, with $m > k$.Now, $n(\bigcup_{i=1}^{k} A(i)) = k$.Thus, by proposition 3, $(B, \bigcup_{i=1}^{k} A(i)) \in I(\mathfrak{R})$. By transitivity of $\mathfrak{R}$, the observation that $\{A(1),\ldots,A(j)\} \in F(\bigcup_{i=1}^{j} A(i))$ for all $j \in \{1,..,m\}$ and repeated application of Proposition 5, we get $(A, \bigcup_{i=1}^{k} A(i)) \in P(\mathfrak{R})$. Thus by transitivity of $\mathfrak{R}$ we get $(A,B) \in P(\mathfrak{R})$. Q.E.D.

Proposition 6: $\mathfrak{R}^S$ satisfies INS,SM,SSM,WSC,SC.

Proof: Easy.

Proof of Theorem 2: Follows from Propositions 3,5 and 6.

Proof of Theorem 1:Follows from Claim 1,Theorem 1 and Proposition 6.

References:

1. P.K.Pattanaik and Y.Xu (2000) : "On Diversity and Freedom of Choice", Mathematical Social Sciences 40, 123-130.