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A NECESSARY AND SUFFICIENT CONDITION
FOR A MATRIX TO BE TOTALLY UNIMODULAR

by

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To

Chairman (Research)
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Title of the report A necessary and sufficient condition for a matrix to be totally unimodular

Name of the Author M. Raghavachari

Under which area do you like to be classified? Operations Research

ABSTRACT (within 250 words)

A Characterization of totally unimodular matrices is given in the paper. This provides an iterative method directly applicable on the matrix itself to recognize total unimodularity or otherwise of any given matrix.

Please indicate restrictions if any that the author wishes to place upon this note none

Date 15/9/73

Signature of the Author

[signature]

A NECESSARY AND SUFFICIENT CONDITION FOR A MATRIX TO BE TOTALLY UNIMODULAR

Introduction and Summary

A matrix $C = (c_{ij})$ is said to be totally unimodular if the determinant of every minor of C equals 0, +1 or -1. Some of the classical examples of totally unimodular matrices are the coefficient matrix of the transportation problem, the node-arc incidence matrix of a graph and the node-edge incidence matrix of a graph with no odd cycles. Some of the properties and characteristics of the totally unimodular matrices have been studied by many authors, e.g. A.J. Hoffman and J.B. Kruskal [2], I. Heller and C.B. Tompkins [1]

characterization of these matrices with utmost two non-zero entries in each column was provided by Heller and Tompkins [1]. A detailed analysis along with some sufficient conditions for a matrix of 0's and 1's to be totally unimodular was given by Hoffman and Kruskal [2]. Hoffman and Kruskal mention in their paper (p.235 and 245) that necessary and sufficient conditions easily applicable on the given matrix itself would be very interesting. This paper provides a necessary and sufficient condition for a matrix C with elements 0, 1 or -1 to be totally unimodular. This condition in turn provides an iterative method directly applicable on the matrix itself to recognize total unimodularity or otherwise of any given matrix.

2. Let C be a matrix with m rows and n columns and with rank p. By a fundamental elementary operation on the matrix C, we mean any of the following operations:

(i) Interchange of any two rows (columns)

(ii) Adding or subtracting one row (column) from another row (column)

(iii) Multiplication of a row (column) by -1.

First we have a simple "necessary" condition for a matrix to be totally unimodular.

<u>Theorem:</u> In order that the matrix C be totally unimodular it is necessary that the matrix C can be reduced by fundamental elementary operations to a matrix with p 1's in the leading diagonal and 0's elsewhere.

<u>Proof:</u> Let C be totally unimodular. By performing fundamental elementary operations, if necessary, we can make the leading element of the matrix equal to 1. By fundamental elementary operations we first make all the elements in the first column (except the leading element) equal to zero and then all the elements in the first row (except the leading element) equal to zero. Delete the first row and the first column of the matrix and we can show that the reduced $(m-1) \times (n-1)$ matrix is also totally unimodular. To see this, suppose that this reduced matrix is not totally unimodular. Then there exists $r \times r$ minor with determinant not equal to $0, \pm 1$. Consider now the $(r+1) \times (r+1)$ minor obtained by adjoining to the $(r \times r)$ minor the corresponding elements of the deleted row and column. The value of this $(r+1) \times (r+1)$ minor is unchanged and since this minor is obtained by the fundamental elementary operations, the same $(r+1) \times (r+1)$ minor in the original matrix C, has for its determinant the same value. This contradicts the fact that C was totally unimodular. Repeat successively the above operations on the reduced matrices and we

finally arrive at the form indicated in the theorem. This completes the proof of the theorem.

The proof for the above theorem indicates sometimes a method (directly applicable on the elements of the matrix) to recognize non-total unimodularity. If in the process of the reduction to the required form we get an entry ± 2 in the reduced matrix, we can conclude that the given matrix is not totally unimodular. The following example illustrates this point.

<u>Example 1</u>

A 10 x 10 non-totally unimodular matrix recognized by the above method.

$$\begin{matrix}
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & -1 \\
1 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\
-1 & -1 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 \\
-1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & -1 & 0
\end{matrix}$$

The above theorem provides only a necessary condition for a matrix C to be totally unimodular. The condition is not sufficient as can be seen by the following example:

Example 2:

$$C = \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

This matrix is not totally unimodular but can be reduced to the form indicated in the theorem. The author is indebted for Professor E. Balas for the above example.

We give next a necessary and sufficient condition for a matrix C to be totally unimodular. This condition is easy to state and yields an iterative method directly applicable on the elements of the matrix to determine whether a given matrix is totally unimodular or not. Consider the first column of the matrix. If all the elements of the first column are zero, delete this column from the matrix and then consider the second column. Without loss of generality, therefore, assume that not all elements of the first column are zero. Suppose $c_{i1} > 0$. By fundamental elementary operations we can make all c_{j1} with $c_{j1} = \pm 1$ equal to zero. Thendelete this first column and we have a reduced matrix of dimensions mX (n-1). Consider all these reduced matrices obtained by considering separately all the elements c_{kl} with $c_{kl} > 0$. Clearly the number of such reduced matrices equals the number of elements c_{kl} : k = 1, 2, .. m, that are not zero. We have then the following theorem:

Theorem: 2

The given matrix $C = (c_{ij})$ with m rows and n columns is totally unimodular if and only if each of the above-defined reduced matrices are totally unimodular.

Proof: Necessity

Suppose that one of these reduced matrices, say, obtained from the ith row ($c_{i1} > 0$) is not totally unimodular (TU). This implies that there exists a minor of order r whose determinant is not 0 or ± 1. Case 1: If this does not contain the ith row, we can consider, as in the proof of Theorem 1, the $(r + 1) \times (r + 1)$ minor by adjoining the corresponding elements of the ith row and the elements of the first column with leading element c_{i1} and others all zero. Again by the same argument as in the proof of Theorem 1 we would get a contradiction that the given matrix C was not TU. Case 2: If the minor of order r involves the ith row, consideration of the same $r \times r$ minor in the given matrix C suffices to bring out a contradiction that C was not TU.

Sufficiency:

Assume now that each of the reduced matrices is TU but the given matrix C is not TU. Then there exists a submatrix of C whose determinant is not 0, ± 1. Again we distinguish two cases. Case 1: This matrix involves the first column. Then there must be a non-zero element in this column (otherwise the determinant will be zero) say c_{i1}. Then the reduced matrix corresponding to this element will be non TU, a contradiction. Case 2: The matrix does not involve the first column. This determinant will have the same value (neither 0 nor ± 1) in all the reduced matrices contradicting again the fact that every one of them is TU. This completes the proof of the theorem.

REFERENCES

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