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UTILITY THEORY AND PARTICIPATION IN UNFAIR LOTTERIES

by

Nitin R Patel
Marti G Subrahmanyam

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UTILITY THEORY AND PARTICIPATION IN UNFAIR LITTERIES

Nitin R Patel and Marti G Subrahmanyam*

Since the development of the expected utility maxim by von Neumann and Morgenstern, attempts have been made to provide a rational basis for the observed behaviour of individuals under uncertainty. Given reasonable assumptions regarding the form of the utility function for wealth, certain stylised facts are sought to be explained. One such fact, which will concern us in this paper is the willingness of an individual to purchase insurance and lottery tickets. In this connection, it has been pointed out that there is an inconsistency between the standard assumption of diminishing marginal utility and the observed phenomenon of participation by individuals in fair or even unfair lotteries. If the marginal utility of wealth diminishes consistent with a concave utility of wealth function, a fair lottery with equal odds of winning or loosing an identical amount will never be undertaken. The gain in utility from winning will be less than the loss in utility from losing, so that the expected utility from participation in the lottery is negative. However, the purchase of insurance by individuals is consistent with diminishing

*Both from the Indian Institute of Management, Ahmedabad.

marginal utility of wealth.

Two alternative explanations have been offered to explain the observed purchase by individuals of both insurance and unfair lottery tickets. The first one postulates a "specific utility of gambling" which if added to the "negative" expected utility of a lottery yields a positive value. The other approach suggested by Friedman and Savage (1948) hypothesises a utility of wealth function that is initially concave, has a convex segment and becomes concave once more. Thus, there is a range in between where the individual is willing to participate in an unfair lottery. While both these explanations succeed in rationalising observed behaviour without jettisoning the expected utility maxim, they do not accord with the thoory of consumer choice. In this theory, the utility function in commodity space is derived from the axioms of comparability, transitivity, increasing utility and diminishing marginal utility.

This paper provides an alternative explanation for the observed behaviour of individuals, that is consistent with consumer choice theory. The basic hypothesis is that there are certain commodities, mainly luxuries, which are available only in integer amounts. Further, the price of even a single unit of these commodities is large in relation to an individual's budget constraint, with the result that consumption of these commodities implies sacrifice of consumption of other more basic commodities. This "lumpiness" of high value commodities creates a situation where an individual is willing to participate in unfair gambles and yet is behaving consistently with the expected utility maxims.

We will first demonstrate that an individual with a monotone increasing, concave utility function in commodity space will not participate in an unfair gamble if the commodities are perfectly divisible.

<u>Theorem:</u> An individual with a monotone increasing concave utility function in commodity space who acts so as to maximize expected utility will not participate in unfair gambles if the commodities are perfectly divisible.

We assume that:

1. There are n commodities indexed 1,2,.........n.

2. $$C = \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{bmatrix}$$ is the consumption vector, where C_i is amount of commodity i that is consumed, i = 1,2,......n.

3. $P = (P_1, P_2, \ldots\ldots P_n)$ is the vector of prices, where P_i is the price of commodity i, i = 1,2,.....n.

4. $U: R^{n+} \longrightarrow R$ is the utility function of the individual defined on <u>commodity</u> space. Thus $U(C)$ is the utility from consuming C.

5. u: $R^{n+} \longrightarrow R$ is the utility function of the individual defined on <u>wealth</u> space. Thus u(B) is the utility accuring from an amount of wealth equal to B.

6. The following relations connects u and U:

$$u(B) = \text{Max. } U(C)$$

$$\text{S.t. } P.C \leq B$$

$$C \geq 0$$

<u>Proof</u>

We shall first show that u(B) is a monotone increasing concave function of B. Let B_1, B_2 ($B_1 \neq B_2$) be two values of wealth and let the corresponding optimal consumption bundles be C^1 and C^2.

Let $\lambda \in [0, 1]$.

By definition

$$u(\lambda B_1 + (1-\lambda) B_2) = \text{Max } U(C)$$

$$\text{S.t. } P.C. \leq \lambda B_1 + (1-\lambda) B_2, \quad C \geq 0$$

Now $C = \lambda C_1 + (1-\lambda)C_2$ is a feasible consumption bundle for this budget, since $P(\lambda C_1 + (1-\lambda)C_2) = \lambda PC_1 + (1-\lambda)PC_2 = \lambda B_1 + (1-\lambda)B_2$

so that

$$u(\lambda B_1 + (1-\lambda)B_2) \geq U(\lambda C_1 + (1-\lambda)C_2) \quad (1)$$

From the concavity of U it follows that

$$U(\lambda C_1 + (1-\lambda)C_2) \geq \lambda U(C_1) + (1-\lambda)U(C_2) \quad (2)$$

Further

$$\lambda U(C_1) + (1-\lambda) U(C_2) = \lambda u(B_1) + (1-\lambda)u(B_2) \qquad (3)$$

from the optimality of C_1 and C_2.

From (1), (2) and (3), it follows that

$$u(\lambda B_1 + (1-\lambda)B_2) \geqslant \lambda u(B_1) + (1-\lambda) u(B_2)$$

so that u(B) is a concave function of B.

Also, it is obvious that $u(B_1) > u(B_2)$ for all $B_1 > B_2$ from the fact that any feasible commodity bundle with a budget of B_2 is also feasible for the larger budget B_1. Hence, U(B) is a monotone increasing function.

Consider a gamble which will result in outcomes of wealth amounting to W_1, W_2.......W_m with probabilities q_1, q_2.......q_m, $q_i \geqslant 0$ for all i and $\sum_{i=1}^{m} q_i = 1$. Let the gamble be unfair i.e. $\sum_{i=1}^{m} q_i W_i < W$ where W is the amount of wealth possessed by the individual before participating in the gamble. If he maximizes expected utility, he will participate in the gamble if and only if $u(W) \leqslant \sum_{i=1}^{m} q_i u(W_i)$.

From the monotone increasing property of u, it follows that

$$u(W) > u\left(\sum_{i=1}^{m} q_i W_i\right) \qquad (4)$$

From the concavity of u, it follows that

$$u\left(\sum_{i=1}^{m} q_i W_i\right) \geqslant \sum_{i=1}^{m} q_i\, u(W_i) \tag{5}$$

$$u(W) > \sum_{i=1}^{m} q_i\, u(W_i) \tag{6}$$

Thus he does not participate in the gamble.

(Q.E.D.)

The key assumption in the above proof was regarding the divisibility of all commodities. Once we introduce lumpiness of some commodities, it is no longer possible to assert the rejection of all unfair gambles.

Theorem: An individual with a monotone increasing concave utility function on commodity space who acts to maximize expected utility may choose to participate in certain unfair gambles provided at least one commodity is lumpy.

To prove this it will suffice to take the simplest case where we have an additive utility function and only two commodities available for consumption. We shall consider the situation with one commodity being perfectly divisible and the other being lumpy. To avoid getting into irrelevant complexities we shall further assume that the lumpy commodity is available only in quantities of zero or one unit.

Let us suppose commodity 1 is indivisible and only 0 or 1 units of it can be consumed. Let us denote the additive utility function by $u_1(c_1) + u_2(c_2)$. Where u_1, u_2 are strictly concave, continuously differentiable, monotone increasing functions.

To simplify our notation we will assume:

1. Scale and origin for u_1 and u_2 are chosen such that $u_1(0) = u_2(0) = 0$ and $u_1(1) = 1$
2. Units of the commodities are chosen so that the price of each commodity is unity.
3. We define $u(c_2) \equiv u_2(c_2)$. (To avoid carrying along a cumbersome subscript)

Now let us construct the utility function $v(B)$ as a function of the wealth B. For $B \not< 1$, the person has no choice but to consume as much commodity 2 as he can. Thus,

$$v(B) = u(B), \quad B < 1.$$

For $B \geq 1$, he can choose between consuming 0 or 1 unit of commodity 1(spending the remaining wealth on commodity 2).

Thus, $v(B) = \text{Max}\left\{u(B),\ 1+u(B-1)\right\} \qquad B \geq 1$

There are two distinct cases:

i. $u(1) < 1$. This case implies a discontinuity in $u(B)$ at $B = 1$.

ii. $u(1) \geq 1$ This case implies continuity of $u(B)$

In each case we will show that there will be unfair lotteries which will be attractive to an expected utility maximizer whose wealth is in a certain range.

<u>Case</u> (i) $(u(1) < 1)$

To treat case (i) we need to distinguish **two sub-cases,** namely:

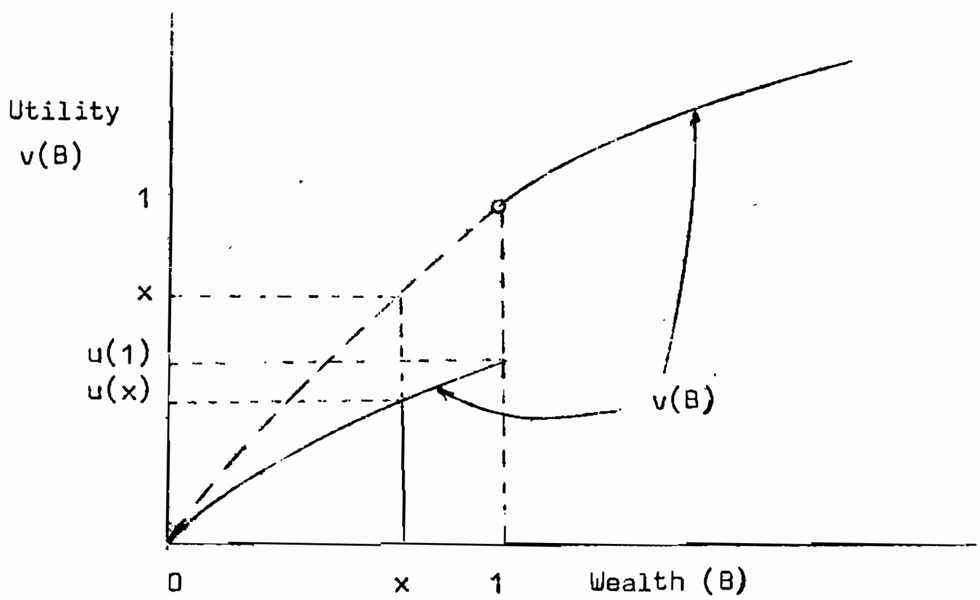
(i/i) $u'(0) \leqslant 1$

(i/ii) $u'(0) > 1$

<u>Sub-case (i/i)</u> $(u(i) < 1,\ u'(0) \leqslant 1)$

Figure 1 depicts v(B) in this sub-case.

<u>Figure 1</u>



We notice in this sub-case that the line joining (0,0) to (1,1) is above u(B) (and hence v(B)) for $0 \leq B < 1$. Hence $u(x) < x$ for $x \in (0, 1)$

We will show that if the individuals wealth is between 0 and 1 he will participate in unfair lotteries. Let his wealth be x ($0 < x < 1$)

Consider the following lottery:

Probability of winning $= \frac{x + u(x)}{2}$

Amount gained from winning = 1 - x

Probability of losing $= 1 - \frac{x + u(x)}{2}$

Amount lost from losing = x

This is an unfair lottery, since its expected value is

$$(1-x)\left\{\frac{x+u(x)}{2}\right\} + (-x)\left\{1 - \frac{x + u(x)}{2}\right\}$$

$$= \left\{\frac{x + u(x)}{2}\right\} - x = \frac{u(x) - x}{2} < 0 \quad \text{since } x > u(x) \text{ for } x \text{ in } (0,1) \qquad (7)$$

However, the expected utility of the lottery will be

$$\left(\frac{x + u(x)}{2}\right) 1 + 0 > u(x) \qquad (8)$$

where the right hand side represents the utility of not participating in the gamble.

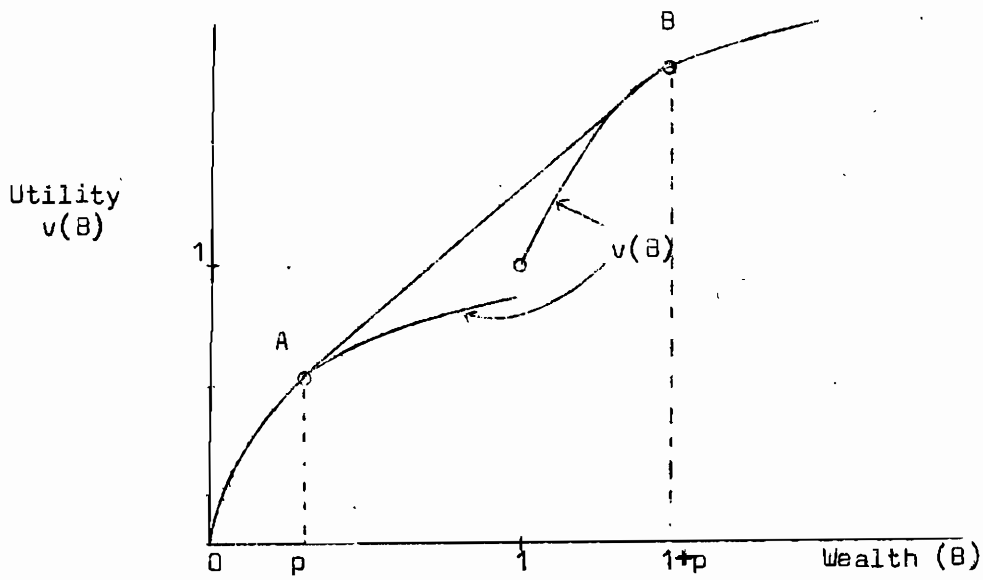
<u>Sub-case</u> (i/ii) ($u(1) < 1$, $u'(0) > 1$)

In this case $u'(1) < 1$. For, if not, $u'(x) \geq 1$ for all x in (0,1) and $u(1) = \int_0^1 u'(x)dx \geq \int_0^1 1.dx = 1$ which contradicts the hypothesis of case (i)

From the assumption that u is continuously differentiable, and $u'(0) > 1$, $u'(1) < 1$ it follows that there is a $p \in (0,1)$ with $u'(p) = 1 = v'(p)$. Also, since for $B \geq 1$, $v(B) = 1 + u(B-1)$ it follows that at $1 + p$, the slope of v is $u'(p) = 1$.

For this sub-case, the nature of v is depicted in **Figure** 2.

Figure 2



We will show that the line passing through $A(p, u(p))$ and $B(1 + p, 1 + u(p))$ is a common tangent to v at A and B. This follows directly from the fact that the slope of the line $= 1 = v'(p) = v'(1+p)$. The equation of this line is $y = x + u(p) - p$ and the line segment AB lies entirely above $v(x)$ for $x \in (p, 1+p)$.

Now, if the individual's wealth = x, $x \in (p, 1 + p)$ again we can construct an unfair lottery that will be attractive to the individual.

To see this consider the following lottery:

Probability of winning = $\dfrac{x + v(x) - u(p) - p}{2}$

Amount gained from winning = $1 + p - x$

Probability of losing = $1 - \dfrac{x + v(x) - u(p) - p}{2}$

Amount lost from losing = $x - p$

The lottery, is unfair, as its expected value is given by:

$$\frac{(x + v(x) - u(p) - p)}{2}(1 + p - x) + \left\{1 - \frac{x + v(x) - u(p) - p}{2}\right\}(p - x) \qquad (9)$$

$$= \frac{v(x) - (x + u(p) - p)}{2} < 0$$

since $x + u(p) - p$ is a point on the line segment AB

The expected utility of this lottery

$$= \frac{(x + v(x) - u(p) - p)}{2}u(1+p) + \left\{1 - \frac{x+v(x)-u(p)-p}{2}\right\}u(p)$$

$$= \frac{(x + v(x) - u(p) - p)}{2}(1+u(p)) + \left\{1 - \frac{x+v(x)-u(p)-p}{2}\right\}u(p)$$

$$= \frac{x + u(p) - p + v(x)}{2} \qquad (10)$$

But $x + u(p) - p$ is a point on the line segment AB and is greater than $v(x)$, so that the expected utility from participating in the lottery $> v(x)$ the utility from not participating. Hence he will participate in this lottery, which is unfair.

<u>Case (ii)</u> ($u(1) \geq 1$)

In this case, the function $v(x)$ has no discontinuity at $x = 1$, since

$$v(x) = \begin{cases} u(x) & 0 \leq x < 1+h \\ 1 + u(x-1) & x \geq 1+h \end{cases}$$

where $h\ (> 0)$ is the solution to $u(1+h) = 1 + u(h)$.

We first note that $u'(0) > 1$. Since $u(1) =$

$$\int_0^1 u'(x)dx < \int_0^1 u'(0)dx = u'(0) \qquad (11)$$

it follows that $u'(0) > u(1) \geq 1$ (12)

Also, we can show that $u'(1+h) < 1$ from the following argument:

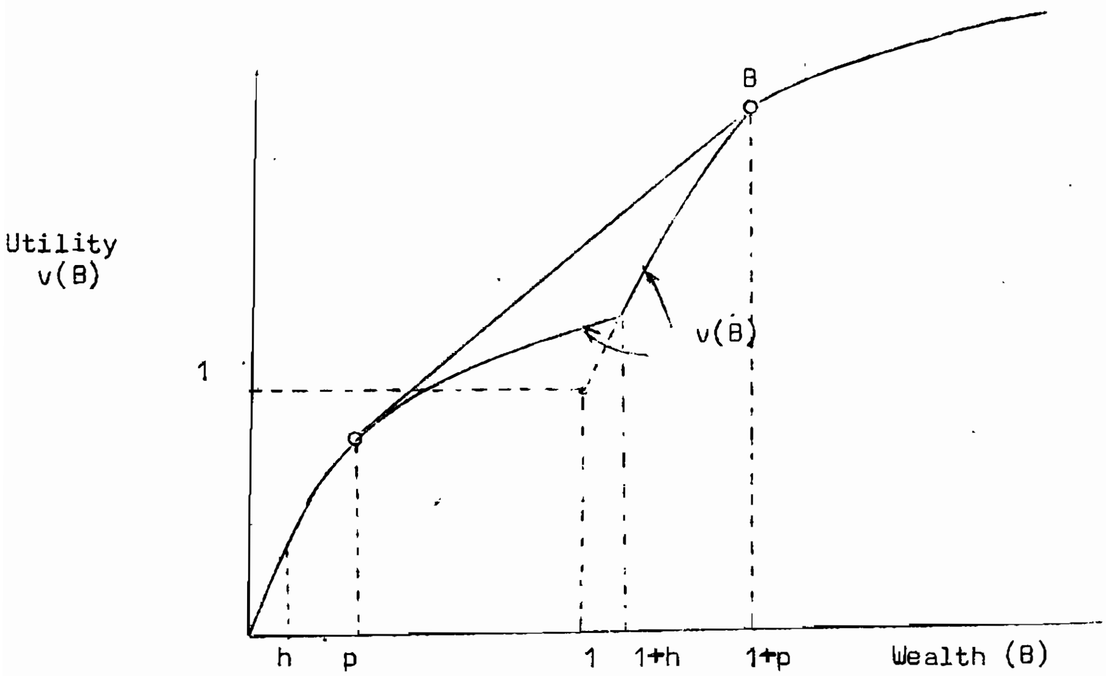
$$u(1+h) = u(h) + \int_h^{1+h} u'(x)\, dx \Rightarrow \int_h^{1+h} u'(x)dx = u(1+h)-u(h) = 1 \qquad (13)$$

From the concavity of u, $u'(x) > u'(1+h)$ for $x < 1+h$, so that

$$u'(1+h) < \int_h^{1+h} u'(x)dx = 1 \qquad (14)$$

From (13) and (14) and the continuous differentiability of u, there must be a point $p \in (h, 1+h)$ with $u'(p) = 1$. Since $p > h$, it follows that $v(1+p) = 1 + u(p)$ and $v'(1+p) = u'(p) = 1$. The solution is diagrammed in Figure 3.

Figure 3



Again we have a situation where the line segment connecting A(p, u(p)) to B (1+p, 1+u(p)) is the common tangent at A and B to **v.** Using exactly the same argument of case (i) sub-case (i/ii) we can show that if the individual has wealth x, $x \in (p, 1+p)$ he will be willing to gamble on unfair lotteries.

We have thus shown that in all cases, there are unfair lotteries. which the individual will participate in, if his wealth falls in certain ranges.

(Q.E.D)

The propensity to participate in unfair gambles is diminished when there is a possibility of saving and consuming the lumpy commodity. To illustrate this consider an individual with a two period utility function which is additive across commodities and across time $U_1(C_1^A) + U_1(C_1^B) + U_2(C_2^A) + U_2(C_2^B)$ where subscripts A and B indicate the two periods. Making the same assumptions and using the same notation as in the previous analysis, consider case (i), sub case (i/i), where the individual has an income $x + \frac{1}{2}$ each in the two periods, where $0 \leq x < \frac{1}{2}$. If the individual saves and does not participate in the gamble his utility is given by

$$\text{Max}\left\{ 2\,u(x + \tfrac{1}{2}),\ 1 + 2\,u(x) \right\}$$

From the concavity of u(.), it follows that

$$u(x) + u'(x) \cdot \tfrac{1}{2} > u\,(x + \tfrac{1}{2}) \tag{15}$$

Since $u'(0) \leq 1$ and $u''(.) < 0$, $u'(x) < 1$, so that

$$u(x) + \tfrac{1}{2} > u(x + \tfrac{1}{2}) \tag{16}$$

which implies that the utility of saving = $1 + 2\,u(x)$

If the individual participates in the lottery defined earlier, assuming that the interest rate is zero, expected utility of gambling

$$= \frac{x + u(x)}{2}\left\{1 + 2u(x) + \tfrac{1}{2}\right\} + \left\{1 - \frac{x + u(x)}{2}\right\}\left\{2\,u(x + \tfrac{1}{2})\right\} \tag{17}$$

Incremental utility of saving = $1 + 2u(x) - 2\,u\,(x + \tfrac{1}{2})$

$$- \frac{(x + u(x))}{2} \tag{18}$$

From the concavity of u(.),

$$u(x) + u'(x)\left(\tfrac{1}{2} - \tfrac{x}{2}\right) > u\,\frac{(x+1)}{2}$$

or $u(x) + \left(\tfrac{1}{2} - \tfrac{x}{2}\right) > u\,\frac{(x+1)}{2}$ (19)

since $u'(x) < 1$, as $u'(0) < 1$ and $u''(.) < 0$

Since $x > u(x)$ in this case, it follows from the above inequality that it is better to save than to participate in the unfair gamble that was previously attractive.

CONCLUSION

This paper has demonstrated that a basic explanation can be provided for the commonly observed phenomenon of participation in unfair lotteries. We provide an explanation that is consistent both with consumer choice theory and the von Neumann-Morganstern axioms, and results from the lumpiness of many commodities, which are available only in integer amounts. In this situation, it is reasonable to participate in a lottery which provides a greater chance of purchasing the lumpy commodity. The expected utility of participation in the lottery exceeds the utility of consumption of more basic commodities which are divisible. In cases where saving is possible, gambling becomes less attractive since the option of buying the lumpy commodity through accumulation of savings over time becomes attractive.

Reference

1. Friedman. M and L. J. Savage, "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy*, Vol. LVI, No.4, August 1948, pp. 279-304.